Adiabatic quantum pumping in open and closed coherent systems

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I. THE VARIETY OF PUMPING MECHANISMS

Pumping one can define as a phenomenon when the dc current is generated in the system with the help of only local perturbation, without applying a global driving (or bias) Fig.1. A variety of different pumping mechanisms in mesoscopic systems are considered in the literature. I give only a few examples.
\[ U(t, \theta) = U_0(\theta + \omega t) \]

\[ U(t, x) = U_0(x - vt) \]

FIG. 2: Traveling potential: a) closed system; b) open system.

1. **Thouless pump**

Probably Thouless was the first who considered the effect of a traveling periodic potential Fig. 2 on electron system.¹ (Such a potential drags electrons and, thus, generates a current.)

Thouless showed that the charge pumped for a cycle can be arbitrary (*not quantized*) for a closed system, and it is *quantized* (in units of elementary charge “e”) for an open system¹,² if there is a gap in electron spectrum (if any band is either full or empty).
2. Acoustoelectric pumping

The straightforward development of Thouless’s ideas (that was realized experimentally already) is a pump driven via surface acoustic waves propagating through the 2D electron gas\textsuperscript{3–5}. These waves induces moving potential wells which can trap one or several electrons and carry them from one bank to another bank Fig.3.

Since each trap contains an integer number of electrons - the pumped charge, in principle, is obviously quantized.
Another possibility (which also was realized) to achieve a quantized pumping is to exploit the Coulomb blockade effect\textsuperscript{6–9}. In this case Fig. 4 the potentials responsible for electron jumps from/to the left (right) contact to/from the sample are varied in such way to enforce electrons move to one way (e.g., from the left to the right) only.
4. Pumping as a result of a photovoltaic effect

The irradiation of a sample without inversion center by electromagnetic waves can cause a direct current flow. In a mesoscopic sample a random impurity distribution breaks the inversion symmetry and thus can lead to a photovoltaic effect\textsuperscript{10}. Therefore at some conditions an electromagnetic radiation (EMR) can induce a current flowing in a random, sample-specific direction Fig.5. The EMR illumination creates a nonequilibrium electron distribution. And the relaxation owing to inelastic interactions leads to a current\textsuperscript{11}. (Electron wave functions at different energies have an effective center of mass at different positions. Therefore when an electron changes its energy it in turn changes its position, that means a current).

Note that this mechanism does not produce a time-average dc current $\langle I(t) \rangle = 0$. The only noise $\langle I^2(t) \rangle \neq 0$ is produced.
More controllable (compared with a photovoltaic effect) pumping can be achieved in mesoscopic systems with quantized spectrum, e.g., two coupled quantum dots. If, in addition, one dot is coupled to, say, left lead and the other dot is coupled to the right lead and the radiation has a frequency $\omega$ equal to the difference in dots levels $\hbar \omega = E_1 - E_2$ then a dc current can be generated Fig. 6.

This current is *not quantized* any more.

This example gives essential what is characteristic for, so called, *nonadiabatic pumping*.
FIG. 7: A quantum dot with scattering matrix $\hat{S}$ and two leads. Two nearby metallic gates modulate the shape and hence the scattering properties of the dot. If the gate potentials $V_1$ and $V_2$ change cyclically but shifted in phase then a current $I_{dc}$ can arise in the leads. (a) - in an open conductor the current $I_{dc}$ flows between the external reservoirs; (b) - in a closed conductor the current $I_{dc}$ flows along a ring of length $L$ formed by the leads. The Greek letter $\alpha$ numbers the scattering channels.

6. Adiabatic quantum pumping

Let us consider a mesoscopic sample (scatterer) coupled to reservoirs or embedded into a ring Fig.7. Varying slowly scattering properties of a sample (i.e., the parameters $P_i$, $i = 1, 2, \ldots$ which determine the scattering properties) one can generate a dc current in a system. If only a single parameter $P_1(t) = P_1 \cos(\omega t)$ oscillates then a current can not be generated. In contrast two potentials oscillating with the same frequency but out of phase

$$P_1(t) = P_1 \cos(\omega t + \varphi_1),
\quad P_2(t) = P_2 \cos(\omega t + \varphi_2),
\quad \Delta \varphi = \varphi_2 - \varphi_1 \neq 0$$

can generate a dc-current

$$I_{dc} \sim \sin(\Delta \varphi).$$

The effect is of interest under conditions in which electron motion is phase-coherent through the system and is thus termed *quantum* pumping. The frequency of the potential modulation is small compared to the characteristic times for traversal and reflection of electrons and the pump is thus termed *adiabatic*. Further we give different (but equivalent) definition for the adiabaticity.

This current is, in general, *not quantized* however at some conditions it can be *quantized*.

This pump is a subject of the present talk.
From above mentioned examples one can extract, at least, two basic pumping principles: classical and quantum.

A. Classical (incoherent) pumping

Classical pumping consists in the following: We localize electron in some spatial region and then enforce it to move in predefined direction. The scheme is depicted in Fig.8.

There are three gates creating a left potential barrier $V_L$, a right barrier $V_R$, and a bottom of a potential well $V_G$, respectively. Manipulating by these gates one can push some amount of a charge $\delta Q$, say, from the left to the right. If the potentials $V_L$ and $V_R$ increase to infinity then the pumped charge is quantized $\delta Q = Ne$. It is because during the stage ”four” we have a fully decoupled piece of a wire which, apparently, contains an integer number of particles.

Therefore one can achieve quantized pumping.

Such pump pushes the same amount of charge per cycle no matter either it is coupled to reservoirs or it is embedded in the ring.

If one removes the stage ”one” then one gets the pump which is fully reflecting

$$R = 1,$$

during the whole cycle.

The important feature of this pumping principle is breaking of a global coherence (through the entire system): see, at least the stage ”four”.

Note that the pump depicted in Fig.8 can be represented in another but, in principle, equivalent manner. If one uses a moving potential wall Fig.9 then one gets ”snowplow” pumping picture given by Avron et al.\textsuperscript{16}
A classical pump

FIG. 8: Classical pumping.
The snowplow
(J.E. Avron, A. Elgart, G.M. Graf, L. Sadun)

1) 

2) 

3) 

4) 

5) 

FIG. 9: Avron’s et al. "snowplow" pump.
However there exists a possibility to produce directed electron flows without destroying phase coherence. To generate such flows one needs to break the symmetry between the movement to the left and the movement to the right.

Strictly speaking, an adiabatic quantum pump effect consists in the possibility for nonstationary scatterer to break such a left-right symmetry.

This, quantum, principle exploits peculiarities of scattering by oscillating scatterer (e.g., a potential barrier) Fig.10. The main difference between scattering by the stationary potential barrier and scattering by nonstationary (oscillating) potential barrier consists in arising side bands in the last case Fig.10: The scattered particle can gain or loss some modulation quanta $n \hbar \omega$ and leave the scattering region with energy $E_n = E + n \hbar \omega$ different from an incoming energy $E$.

This evident fact underlies the adiabatic quantum pump effect.

In open (i.e., connected to external reservoirs) and closed (i.e., isolated) systems this inelastic scattering manifests itself in slightly different way.
stationary scattering:  \[ U = U_0 \]

periodically driven scattering:  \[ U = U_0 + U_1 \cos(\omega t) \]

FIG. 10: Scattering problem: a) stationary barrier; b) oscillating barrier.
An open system has a continuous spectrum. Therefore absorbing energy quantum $\hbar \omega$ produces interlevel transitions. Because of the Pauli principle an electron with energy $E$ can absorb an energy quantum $\hbar \omega$ if only the state with energy $E + \hbar \omega$ is empty. On the other hand, after absorbing the state with energy $E$ becomes empty.

Therefore the absorption of an energy quantum $\hbar \omega$ can be viewed as a creation of a nonequilibrium quasi-electron hole pair.

After being created these quasi particles can leave the scattering region through the different leads Fig.11. In this case they do contribute into the charge transfer between the corresponding reservoirs.

Averaging over many statistically independent of each other events of creation quasi particle pairs one can calculate the net current flowing in any leads.

To calculate the (dc) current $I_\alpha$ flowing in the lead $\alpha$ one needs to calculate the distribution function for quasi-electrons $N^{(e)}_\alpha$ and holes $N^{(h)}_\alpha$ leaving the scatterer through the lead $\alpha$. Then the current reads as follows: $I_\alpha = eN^{(e)}_\alpha + (-e)N^{(h)}_\alpha$, where $e(-e)$ is the charge of an electron (a hole).

If the time reversal invariance (TRI) is broken by the oscillating scatterer: $\hat{S}(t) \neq \hat{S}(-t)$, then the probability for a quasi-electron to leave the scatterer through some leads, say $\alpha$, is different from the corresponding probability for a hole. Such a difference in probabilities leads to the difference in the distribution functions that in turn results in a dc current.

This is a reason for a pump effect in open coherent systems.

In terms of real particles one can reformulate above as follows: In open system the breaking of the Time Reversal Invariance leads to nonuniform redistribution of incoming fluctuating flows between outgoing channels (leads).
A quantum (coherent) pump: an open system

many particle system
with continuous spectrum:

\[ \Sigma n>0 \Sigma n h \omega_{\alpha n} \]

\[ N = \sum |S_{\alpha n}|^2 \]

\[ N^{(h)} = \sum |S_{\alpha n}|^2 \]

A broken TRS: \( \tilde{S}(t) \neq \tilde{S}(-t) \Rightarrow S_{\alpha n} \neq S_{\alpha n} \]

\[ I_{dc, \alpha} = eN^{(e)} + (-e)N^{(h)} \neq 0 \]

FIG. 11: Pumping in open case.
In closed systems Fig.12 the energy absorption leads to a completely different picture. The closed system has, in general, a discrete spectrum. Therefore in an adiabatic case the true quantum-mechanical adiabaticity condition can be achieved
\[ \hbar \omega \ll \Delta, \] 
where \( \Delta \) is the level spacing.

In this case the energy exchange with an oscillating scatterer does not produce any interlevel transitions.

However such exchange (according to the Floquet theorem) does split any energy level \( E \) into an energy ladder \( E_n = E + n \hbar \omega \), where \( n = 0, \pm 1, \pm 2, \ldots \).

If the pump parameters oscillate with small amplitudes \( P_i(t) = P_{i,0} + P_{i,1} \cos(\omega t + \varphi_i), P_{i,1} \ll P_{i,0} \) then the energy level \( E_0 \) is the same as in stationary \( (P_{i,1} = 0) \) case. The energy \( E \) is the solution for the corresponding dispersion equation and the corresponding eigen wave function is subject to the constructive interference in a ring. In contrast the side bands \( (n \neq 0) \) are always subject to the destructive interference in a ring. Therefore the intensities of side bands are small. Such intensities only a little depends on the pump intensity (in contrast, in open case the intensity of side bands depend strongly on the pump intensity).

If the pump parameters oscillate with large amplitudes then each eigen energy becomes a function of the time: \( E = E(t) \).

If the scatterer breaks the Time Reversal Invariance then each energy level (as main component as side bands) carry some current. In other words: In stationary case or if the TRI is present then the wave functions are ”standing waves”. Whereas if the TRI is broken then the wave functions become ”running”.

C. Real pumping

It should be noted that any real pumping mechanism involves partially both the classical and the quantum pumping principles.
A quantum (coherent) pump: a closed system

An adiabatic limit in systems with discrete spectrum: $\hbar \omega \ll \Delta E$

<table>
<thead>
<tr>
<th>Stationary System</th>
<th>Periodically Driven System (Weak Pumping)</th>
<th>Periodically Driven System (Weak Pumping, Broken TRS)</th>
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<tbody>
<tr>
<td>$E$</td>
<td>$E + \hbar \omega$</td>
<td>$E + \hbar \omega$</td>
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<tr>
<td>Standing Wave</td>
<td>$E$</td>
<td>$E$</td>
</tr>
<tr>
<td></td>
<td>$E - \hbar \omega$</td>
<td>$E - \hbar \omega$</td>
</tr>
</tbody>
</table>

Standing waves:

Running waves:

$I_\Sigma = I_{(0)} + I_{(+)} + I_{(-)} \sim \omega \sin(\varphi)$

Periodically Driven System

(Strong Pumping, No Level Crossings, Broken TRS)

Running waves:

$I_\Sigma \sim \omega$; $I_\Sigma (\varphi) = -I_\Sigma (-\varphi)$

FIG. 12: Pumping in closed case.
Now we consider in more detail adiabatic quantum pumping. We give a description within the framework of the Floquet scattering theory.

Let us consider a multiterminal scatterer coupled to several $N_r$ reservoirs. We will consider non-interacting spinless fermions and suppose the leads connecting the scatterer to the reservoirs to be single-channel and ballistic Fig.13.

Let us introduce a wave function $\psi^{(in)}_\alpha$ for incoming state in the lead $\alpha$ and a wave function $\psi^{(out)}_\alpha$ for outgoing state in the lead $\alpha$. It is convenient to unite all incoming and outgoing wave functions into the vector-columns:

$$
\hat{\Psi}^{(in)} = \begin{pmatrix} 
\psi^{(in)}_1 \\
\psi^{(in)}_2 \\
\vdots \\
\psi^{(in)}_{N_r}
\end{pmatrix},
\hat{\Psi}^{(out)} = \begin{pmatrix} 
\psi^{(out)}_1 \\
\psi^{(out)}_2 \\
\vdots \\
\psi^{(out)}_{N_r}
\end{pmatrix}.
$$ (5)

Then we can introduce a scattering matrix which relates the incoming states to outgoing ones. In a general nonstationary case a scattering problem is enough complicated.

However if the scattering properties of a sample changes periodically in time with period $T = \frac{2\pi}{\omega}$ then the nonstationary problem can be reduced to the stationary one.

Remember that stationary scattering is elastic (i.e., the energy of outgoing particles coincides with the energy of incoming particles). Therefore the stationary scattering matrix $\hat{S}(E)$ is of dimension $N_r \times N_r$ and it relates the incoming states to outgoing ones (both given at some fixed energy $E$):

$$
\hat{\Psi}^{(out)}(E) = \hat{S}(E)\hat{\Psi}^{(in)}(E).
$$ (6)

To get the elements $S_{\alpha\beta}$ of the scattering matrix one needs to solve the stationary Schrödinger equation with corresponding boundary conditions. One can say that the above equation is a convenient way to write down the solution of the stationary scattering problem. Once solved scattering problem with unit incoming waves can be used to find the amplitudes of outgoing waves at arbitrary incoming waves.

Let us consider the scattering of incoming wave $\psi^{(in)}_\alpha(E)$ with definite energy $E$ by the periodically oscillating scatterer. According to the Floquet theorem the full set $\{E_n\}$ of possible energies for outgoing particles is as follows:

$$
E_n = E + n\hbar\omega, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots.
$$ (7)

Therefore this particular nonstationary problem can be reduced to the stationary problem with only each eigenenergy $E$ being replaced by the corresponding ladder $E_n$.

Such scattering problem can be described via the Floquet scattering matrix $\hat{S}_F(E_n, E)$ depending on two energies, incoming $E$ and outgoing $E_n$:

$$
\hat{\Psi}^{(out)}(E_n) = \hat{S}_F(E_n, E)\hat{\Psi}^{(in)}(E).
$$ (8)
The matrix element $\hat{S}_{F,\alpha\beta}(E_n, E)$ is a quantum-mechanical amplitude for the particle incoming with energy $E$ from the lead $\beta$ to be scattered to the lead $\alpha$ having energy $E_n = E + n\hbar\omega$. If $n > 0$ then an electron absorbs $n$ energy quanta $\hbar\omega$ from the oscillating scatterer. If $n < 0$ then an electron emits $n$ energy quanta $\hbar\omega$ (i.e., an electron gives this energy to the oscillating scatterer).

Now we introduce a notion of adiabatic scattering for open systems (with continuous spectrum) in context of the Floquet scattering theory. The exact quantum-mechanical adiabaticity is always violated in systems with continuous spectrum (excepting some special cases). Therefore we need to introduce another but still reasonable definition of adiabaticity. It seems convenient (and reasonable) to consider a dynamical system as an adiabatic one if their scattering properties are defined entirely by the stationary scattering matrix (with parameters being the functions of time).

In general the Floquet scattering matrix depends on two energies, incoming $E^{(in)}$ and outgoing $E^{(out)} = E^{(in)} + n\hbar\omega$. The Floquet scattering matrix for the adiabatic system has to depend on only one energy. This condition can be satisfied if the stationary scattering matrix for $E = E^{(in)}$ and for $E = E^{(out)}$ could be approximately the same.

Therefore we suppose scattering to be adiabatic if the energy quantum $\hbar\omega$ is small compared with the relevant energy scale $\delta E$ over which the stationary scattering matrix $\hat{S}(E)$ changes significantly:

$$\hbar\omega \ll \delta E.$$  \hspace{1cm} (9)

To clarify the definition of $\delta E$ let us consider the scatterer with transmission resonances Fig.14. Close to the resonance $\delta E$ is of order of the resonance width while in between the resonances $\delta E$ is rather of the order of the distance between the consequent resonances.

For adiabatic pumping one can expand the Floquet scattering matrix in powers of $\epsilon = \frac{\hbar\omega}{\delta E}$.

The lowest (zero order) approximation\(^{17}\) can be expressed in terms of a stationary scattering matrix with time-dependent parameters (the frozen scattering matrix): $\hat{S}(E, t) \equiv \hat{S}(E, \{P(t)\})$. Here $\{P\}$ is a set of parameters $P_i(t) = P_{i,0} + P_{i,1}\cos(\omega t + \varphi_i), i = 1, 2, \ldots, N_p$ oscillating with frequency $\omega$. The scattering matrix $\hat{S}(E, \{P\})$ describes reflection and transmission of particles with energy $E$ at given frozen parameters $P_i$.

We introduce the Fourier coefficients $\hat{S}_n(E)$ for the frozen scattering matrix $\hat{S}(E, t)$,

$$\hat{S}_n(E) = \frac{\omega}{2\pi} \int_0^T dt e^{in\omega t} \hat{S}(E, t).$$ \hspace{1cm} (10a)

$$\hat{S}(E, t) = \sum_{n=-\infty}^{\infty} e^{-in\omega t} \hat{S}_n(E),$$ \hspace{1cm} (10b)

Then the zero order adiabatic Floquet scattering matrix equals to:

$$\hat{S}_F(E_n, E) \approx \hat{S}_F(E, E_{-n}) \approx \hat{S}_n(E).$$ \hspace{1cm} (11)

We use above definition to analyze the conditions which are necessary to have an adiabatic pump effect.
FIG. 13: A mesoscopic pump with scattering matrix $S(t)$ oscillating with frequency $\omega$ is coupled to $N_r$ reservoirs.
FIG. 14: The dependence of the transmission coefficient on the energy. Close to the resonance $\delta E$ is of order of the resonance width while in between the resonances $\delta E$ is rather of the order of the distance between the consequent resonances.
IV. A BROKEN TIME REVERSAL INVARIANCE IS A NECESSARY CONDITION FOR THE ADIABATIC QUANTUM PUMP EFFECT

A. Open systems

The (dc) current pumped in the lead $\alpha$ can be expressed in terms of the Floquet scattering matrix as follows:

$$I_\alpha = \frac{e}{\hbar} \int_0^\infty \, dE \sum_{\alpha} \sum_{E_n > 0} |S_{F,\alpha\beta}(E_n, E)|^2 \left( f^{(in)}_{\beta}(E) - f^{(in)}_{\alpha}(E_n) \right). \tag{12}$$

Using Eq.(11) we obtain the adiabatic dc current:

$$I_{ad,\alpha} = \frac{e\omega}{2\pi} \int_0^\infty \, dE \left( -\frac{\partial f_0(E)}{\partial E} \right) \sum_{\beta} \sum_{n=1}^\infty \left( |S_{\alpha\beta,n}(E)|^2 - |S_{\alpha\beta,-n}(E)|^2 \right). \tag{13}$$

Note that above expression is valid as at low $k_B T \ll \hbar \omega$ as at high $k_B T \gg \hbar \omega$ temperatures. We can see that the pumped current is not zero if only

$$|S_{\alpha\beta,n}|^2 \neq |S_{\alpha\beta,-n}|^2. \tag{14}$$

To clarify above condition let us look at the Fourier expansion Eq.(10b):

$$\hat{S}(t) = \sum_{n=-\infty}^{\infty} e^{-in\omega t} \hat{S}_n, \tag{15}$$

$$\hat{S}(-t) = \sum_{n=-\infty}^{\infty} e^{-in\omega t} \hat{S}_{-n}.$$

Therefore to satisfy Eq.(14) the time reversal invariance of the scatterer has to be broken:

$$\hat{S}(t) \neq \hat{S}(-t). \tag{16}$$

In particular this can be achieved if two parameters of a scatterer oscillates with the phase lag $\Delta \varphi \neq 0$, Eq.(1):

$$P_1(t) = P_1 \cos(\omega t), \quad P_1(-t) = P_1(t), \quad P_2(t) = P_2 \cos(\omega t + \Delta \varphi), \quad P_2(-t) \neq P_2(t), \quad \text{if } \Delta \varphi \neq 0.$$

This is exactly what was used by Switkes et al. in experiment$^{13}$ to generate a dc current.
B. Closed systems

It was shown\textsuperscript{18} that the same slow oscillating scatterer generates a dc pumped current no matter either it is coupled to reservoirs (an open case) or it is embedded in the ring (a closed case).

The necessary condition for appearance of a dc pumped current in both (open and closed) cases is the same. It consists in breaking of the time reversal invariance Eq.(16). However the magnitude of a pumped current in open and closed cases is different.

To illustrate such a difference let us consider a scatterer with scattering matrix

\[ \hat{S}_0 = e^{i\gamma} \begin{pmatrix} \sqrt{R}e^{-i\theta} & i\sqrt{T} \\ i\sqrt{T} & \sqrt{R}e^{i\theta} \end{pmatrix}, \]

(17)

where \( R \) and \( T \) are the reflection and the transmission probability, respectively: \( R + T = 1 \); the phase \( \theta \) characterizes the asymmetry between the reflection to the left and to the right; the phase \( \gamma \) relates to the change of the overall charge \( \delta Q \) on the scatterer (for instance a dot) via the Friedel sum rule: \( \delta \gamma = \pi \delta Q / e \).

Then the pumped current (at zero temperature: \( T = 0 \)) reads as follows: (open case - Ref.19; closed case - Ref.18)

- Open system: \( \Rightarrow I_{dc}^{(open)} = \frac{e\omega}{4\pi^2} \int_0^T dt R \frac{\partial \theta}{\partial t} \),

- Closed system: \( \Rightarrow I_{dc}^{(l)} = \frac{e\omega}{4\pi} (-1)^l \int_0^T dt \sqrt{\frac{T}{R}} \frac{\partial \theta}{\partial t} \),

\( I_{dc}^{(closed)} = \sum_l I_{dc}^{(l)} \).

Here \( T = 2\pi / \omega \) is the period of a pump cycle.

Stress that \( I_{dc}^{(open)} \) is a full current pumped between the reservoirs (the quantities, \( R \) and \( \theta \), are given at the Fermi energy: \( E = \mu \)). Whereas \( I_{dc}^{(l)} \) is a current carried by some energy level \( E^{(l)} \). The full current \( I_{dc}^{(closed)} \) circulating in a ring is given by the sum over all occupied levels. Note that \( I_{dc}^{(l)} \) was obtained under conditions when \( R \neq 0 \).

Such difference between the magnitudes of a pumped current in open and closed cases is due to an interference in a ring. While in an open case only the transmission coefficients by modulo Eq.(14) are of importance, in a closed case in addition the corresponding phases become relevant.
To illustrate the importance of the scattering phases let us consider another quantity which breaks the time reversal invariance: Namely, we consider a magnetic flux penetrating a one-dimensional ring Fig.15. This magnetic flux results in the Aharonov-Bohm phase and in the well known persistent currents.

On the other hand, as it is well known, using a gauge transformation one can easily remove the Aharonov-Bohm phase from the Schrödinger equation (and correspondingly from the wave function) to the boundary conditions.

In our case the boundary conditions are given by the scattering matrix. Therefore, the following two problems are equivalent:

(i) a ring with enclosed magnetic flux and with embedded scatterer having the equal amplitudes for scattering from the left to the right and vice versa: \( S_{12} = S_{21} \);

(ii) a ring without enclosed magnetic flux and with embedded scatterer having the different amplitudes for scattering from the left to the right and vice versa: \( S_{12} \neq S_{21} \). Because the scattering matrix is unitary \( \hat{S}\hat{S}^\dagger = \hat{I} \), the transmission amplitudes can differ from each other by only the phase factor: \( S_{12} = Se^{i\phi} \), \( S_{21} = Se^{-i\phi} \).

Thus we conclude, if the scatterer embedded in a ring has unequal amplitudes for transmission to the left and to the right then a circulating current arises.

Note, if one couples such a scatterer to the reservoirs then it can NOT produce a directed current. It is because in open case the transmission probabilities (not amplitudes!) matter. Note that they are the same in the case under consideration: \( |S_{12}|^2 = |S_{21}|^2 = |S|^2 \).

It should be noted that above example is considered only as an illustration to show that in the phase coherent case the same scatterer produces the currents of different magnitude depending on setup: closed (ring) or open (wire).

The currents dynamically generated by the pump are different in nature from the currents induced by the magnetic flux: the former are time-dependent and they can exist not only in a closed ring but in an open wire as well.
FIG. 15: a) A quantum dot with symmetric scattering matrix $S_{12} = S_{21} = S$ is embedded in a one-dimensional ring with enclosed magnetic flux $\Phi$. $\psi(x) = \left(A e^{ik(x-L)} + B e^{-ikx}\right) e^{i\phi} \Phi$ is an electron wave function. Here $\phi = 2\pi \Phi/\Phi_0$, and the $\Phi_0 = \frac{\hbar}{e}$ is the magnetic flux quantum.

b) A quantum dot with asymmetric scattering matrix $S_{12} = e^{-i\phi} S$; $S_{21} = e^{i\phi} S$ is embedded in a one-dimensional ring without a magnetic flux $\Phi = 0$.

There is a circulating current $I \sim \sin(\phi)$ the same in both cases.
Now we consider an open case in more detail. Having in mind applications we need to consider what is happened if one connect the pump to another elements of a mesoscopic circuit. Apparently the circuit has its own impedance and the back action to the pump has to be taken into account. This question is still open. However what already understood is that not only dc current but full time dependent current generated by the pump is important. Therefore let me touch on this question.

Strictly speaking the pump effect is a "dynamical" effect. The existence of a pump effect emphasizes an essential difference between a nonstationary scattering problem and a stationary (or even mere "frozen") one. It is accidentally that one can calculate a pumped current (in open case) using a stationary scattering matrix Eq.(13).

In general case to correctly calculate the current flowing through the oscillating scatterer it is necessary to take into account the difference between the Floquet scattering matrix and the stationary scattering matrix, i.e., to calculate the current of the first order in frequency \( \omega \) one needs to know the Floquet scattering matrix with the same accuracy.

For instance, this is the case if a nonstationary scatterer is coupled to nonstationary reservoirs\(^ {23} \).

To show this formally let me give two different equations which calculate the same quantity - a pumped dc current (for the simplicity we put the temperature to be zero: \( T = 0 \); \( \mu \) is the Fermi energy):

\[
I_\alpha = \frac{e\omega}{2\pi} \sum_\beta \sum_n n |S_{F,\alpha\beta} (\mu + n\hbar\omega, \mu)|^2, \quad (19a)
\]

\[
I_\alpha = \frac{e}{\hbar} \int_0^\infty dE \sum_\beta \sum_n f_0(E) \left( |S_{F,\alpha\beta} (E_n, E)|^2 - |S_{F,\beta\alpha} (E_n, E)|^2 \right). \quad (19b)
\]

We suppose that all the reservoirs are at the same equilibrium conditions and, therefore, it is \( f_\text{in} = f_0 \), where \( f_0 \) is the Fermi function. Since Eq.(19a) contains a term \( \omega \) then to calculate the current \( I_\alpha \) of the first order in \( \omega \) one can use the zero order approximation for the Floquet scattering matrix Eq.(11).

In contrast, if we substitute Eq.(11) in the Eq.(19b) we get identically zero. Therefore to use Eq.(19b) one needs to calculate the Floquet scattering matrix with accuracy of \( \omega \). If we use such corrections then Eq.(19b) gives the same result as Eq.(19a). As we mentioned already, such corrections are necessary to consider ac currents flowing through the nonstationary scatterer.

The first order in frequency (or more precisely in \( \epsilon = \hbar \omega \)) corrections to the adiabatic Floquet scattering matrix Eq.(11) can be expressed in terms of the time and energy derivatives of a frozen scattering matrix (i.e., a stationary scattering matrix with time-dependent parameters).\(^ {23} \) Using them we get the current Eq.(19b) as follows:

\[
\frac{dI_\alpha}{dE} = e \mathcal{P} \{ \hat{S}_0; \hat{S}_0^\dagger \}_\alpha = i \frac{e}{2\pi} \left( \frac{\partial \hat{S}_0}{\partial E} \frac{\partial \hat{S}_0^\dagger}{\partial t} - \frac{\partial \hat{S}_0^\dagger}{\partial E} \frac{\partial \hat{S}_0}{\partial t} \right)_{\alpha\alpha}, \quad (20)
\]

The quantity \( dI_\alpha(E,t)/dE \) is an instantaneous (at time \( t \)) spectral current which is pushed by the oscillating scatterer into the lead \( \alpha \).

If the chemical potential \( \mu \) (of the reservoirs) is independent of time then integrating over energy \( E \) and over time \( t \) by parts one can reduce Eq.(20) to Eq.(13). Applying the inverse Fourier transformation we can represent Eq.(13) in the form first obtained by Brouwer\(^ {14} \):

\[
I_\alpha = \frac{i e\omega}{4\pi^2} \int_0^{2\pi/\omega} dt \int_0^\infty dE \left( - \frac{\partial f_0(E)}{\partial E} \right) \left( \frac{\partial \hat{S}_0(E,t)}{\partial t} \hat{S}_0^\dagger(E,t) \right)_{\alpha\alpha}. \quad (21)
\]
One can introduce even more detailed partitioning of pumped currents:

\[
\frac{dI_\alpha(E, t)}{dE} = \sum_\beta \frac{dI_{\alpha\beta}(E, t)}{dE}. \tag{22}
\]

To this end let us consider the time-dependent current \(I_\alpha(t)\) flowing through the scatterer (in the lead \(\alpha\)) in the case when the reservoirs have the different oscillating potentials \(\mu_\alpha(t) \neq \mu_\beta(t)\):

\[
\mu_\alpha(t) = \mu_0 + \delta \mu_\alpha \cos(\omega t + \phi_\alpha), \quad \delta \mu_\alpha \ll \mu_0. \tag{23}
\]

The current \(I_\alpha(t)\) reads as follows:

\[
I_\alpha(t) = \int_0^\infty dE \sum_\beta \left\{ \frac{e}{\hbar} \left[ f_0(E; \mu_\beta(t)) - f_0(E; \mu_\alpha(t)) \right] \left| S_{0,\alpha\beta}(E, t) \right|^2 - e \frac{\partial}{\partial t} \left[ f_0(E; \mu_\beta(t)) \frac{dN_{\alpha\beta}(E, t)}{dE} \right] + f_0(E; \mu_\beta(t)) \frac{dI_{\alpha\beta}(E, t)}{dE} \right\}. \tag{24}
\]

Here we have introduced the partial density of states\(^\text{15}\) of a "frozen" scatterer,

\[
\frac{dN_{\alpha\beta}}{dE} = \frac{i}{4\pi} \left( \frac{\partial S_{0,\alpha\beta}^*}{\partial E} S_{0,\alpha\beta} - S_{0,\alpha\beta}^* \frac{\partial S_{0,\alpha\beta}}{\partial E} \right). \]

These density of states define the charge \(Q(t)\) of a "frozen" scatterer as follows:

\[
Q(t) = e \sum_\alpha \sum_\beta \int_0^\infty dE f_0(E; \mu_\beta(t)) \frac{dN_{\alpha\beta}(E, t)}{dE}. \tag{25}
\]

The quantity \(I_\alpha(t)\) Eq.(24) and \(Q(t)\) Eq.(25) do satisfy the following continuity equation:

\[
\sum_\alpha I_\alpha(t) + \frac{\partial Q(t)}{\partial t} = 0. \tag{26}
\]

The three terms in the curly brackets on the RHS of Eq.(24) can be interpreted as follows. The first term defines the currents flowing under the action of external voltages \(\mu_\beta - \mu_\alpha\) through a "frozen" scatterer. The second one defines currents attributed to an oscillating charge of a "frozen" scatterer. The third term can not be entirely viewed just as a nonadiabatic correction to either the "frozen" conductance \(\left| S_{0,\alpha\beta}(E, t) \right|^2\) nor to the "frozen" density of states \(\frac{dN_{\alpha\beta}(E, t)}{dE}\). It is more naturally to consider it as the ac currents generated by the oscillating scatterer. The ability to generate these ac currents differentiates a nonstationary dynamical scatterer from a merely "frozen" scatterer.
One can say that the scatterer drives the following spectral currents from the lead $\beta$ into the lead $\alpha$:

$$\frac{dI_{\alpha\beta}}{dE} = \frac{e}{\hbar} \mathcal{P}\{\hat{S}_0^\dagger; \hat{S}_0^\dagger\}_{\alpha\alpha} - \mathcal{P}\{\hat{S}_0^\dagger; \hat{S}_0^\dagger\}_{\beta\beta} + N_r \mathcal{P}\{S_{0\alpha\beta}; S_{0\alpha\beta}^\dagger\}. \quad (27)$$

The above spectral currents are subject to the following conservation law: $\sum_{\alpha=1}^{N_r} dI(E,t)_{\alpha\beta}/dE = 0$. This property supports the point of view that these currents arise "inside" the scatterer (they are generated by the nonstationary scatterer) without any external current source.

For illustrative purpose we give the expressions for the instantaneous generated currents for the (two channels) scatterer with scattering matrix Eq.(17):

$$dI_{11}(E,t) \equiv -dI_{21}(E,t) = -\frac{e}{4\pi} \left( \partial(\gamma - \theta) \partial R - \partial(\gamma - \theta) \partial R / \partial t \right), \quad (28a)$$

$$dI_{22}(E,t) \equiv -dI_{12}(E,t) = -\frac{e}{4\pi} \left( \partial(\gamma + \theta) \partial R - \partial(\gamma + \theta) \partial R / \partial t \right). \quad (28b)$$

$$dI_1(E,t) \equiv -dI_2(E,t) = \frac{e}{2\pi} \left( \partial \theta \partial R - \partial \theta / \partial t \partial R / \partial t \right). \quad (28c)$$

We emphasize that the pump (oscillating scatterer) always produces ac currents $I_\alpha(t)$ Eq.(24). And only if we are interesting in a nonzero $I_\alpha \neq 0$ dc current Eq.(20) we have to break the Time Reversal Invariance in the system.

Therefore we conclude that adiabatically and cyclically evolving scatterer acquires a dynamical characteristic, "instantaneous ac currents", in addition to those which are characteristic for the stationary scatterer Fig.16.

The dc pumped current is a rectified ac current generated by the pump.

In conclusion, we have developed the scattering matrix approach to adiabatic quantum pumping in closed and open mesoscopic systems. This formulation permits a direct comparison of pumping in open and closed systems. The physics underlying the adiabatic quantum pump effect in closed systems is very similar to that in open systems coupled to external reservoirs: The breaking of the time reversal invariance is crucial for generating of a dc current by the slow cyclically evolving mesoscopic system.
\[ N(e) = \int dE \frac{dN}{dE} \]

\[ N(e)(t) = \int dE \frac{dN(t)}{dE} \]

\[ I(\omega)(t) = \int dE \frac{dI(\omega)(t)}{dE} \]

FIG. 16: a) A stationary scatterer; b) A dynamical scatterer.
APPENDIX: A CLOSED SYSTEM. HOW TO PUSH A CURRENT

In this Appendix we prove formally the equivalence of two stationary problems: (i) a ring with enclosed magnetic flux and (ii) a ring with asymmetric stationary scatterer.

For the sake of simplicity we consider a one dimensional ring with a stationary scatterer described via the scattering matrix $\hat{S}$ being a $2 \times 2$ unitary matrix:

$$\hat{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}. \quad (A.1)$$

1. A ring with enclosed magnetic flux

As the first example we consider the symmetric $S_{12} = S_{21} = S$ scatterer of a small size $w \ll L$ embedded in a ballistic ring of length $L$ with magnetic flux $\Phi$, Fig.15a.

The solution for the Schrödinger equation out of the scatterer reads as follows:

$$\psi(x) = \left(A e^{ik(x-L)} + Be^{-ikx}\right) e^{2\pi i \frac{\Phi}{h} \frac{x}{L}}. \quad (A.2)$$

The scatterer introduces effective boundary conditions:

$$\begin{cases} 
\psi^{(\text{out})}(x) = S_{11}\psi^{(\text{in})}(x = L) + S_{12}\psi^{(\text{in})}(x = 0) \\
\psi^{(\text{out})}(x = 0) = S_{21}\psi^{(\text{in})}(x = L) + S_{22}\psi^{(\text{in})}(x = 0).
\end{cases} \quad (A.3)$$

Here the incoming and outgoing waves are as follows:

$$\psi^{(\text{in})}(x = 0) = B; \quad \psi^{(\text{in})}(x = L) = Ae^{i\phi},$$

$$\psi^{(\text{out})}(x = 0) = Ae^{-ikL}; \quad \psi^{(\text{out})}(x = L) = Be^{-ikL}e^{i\phi}, \quad (A.4)$$

where we have introduced $\phi = 2\pi \frac{\Phi}{h} \frac{L}{m_e}$.

Substituting Eq.(A.4) in Eq.(A.2) we get the following:

$$\begin{cases} 
AS_{11} - B\left(e^{-ikL} - e^{-i\phi}S_{12}\right) = 0, \\
A\left(e^{-ikL} - e^{i\phi}S_{21}\right) - BS_{22} = 0.
\end{cases} \quad (A.5)$$

To have a nontrivial solution for the coefficients $A \neq 0, B \neq 0$ the determinant of the system of equations Eq.(A.5) has to be zero. Using $S_{12} = S_{21} = S$ we obtain:

$$\Delta = 0 \implies \left(\frac{e^{-ikL}}{S} - e^{-i\phi}\right)\left(\frac{e^{-ikL}}{S} - e^{i\phi}\right) = -\frac{R}{T}. \quad (A.6)$$

Here we have introduced the reflection $R = |S_{11}|^2 = |S_{22}|^2$ and the transmission $T = |S|^2$ probabilities: $R + T = 1$.

The solutions for the dispersion equation (A.6) are the eigen wave numbers $k^{(l)} = k^{(l)}[S], l = 1, 2, \ldots$, i.e., the wave numbers for allowed states in the ring.

The imaginary and the real parts of Eq.(A.6) give:

$$\begin{align*}
\left(Re\left[\frac{e^{-ikL}}{S}\right] - \cos(\phi)\right)^2 - \left(Im\left[\frac{e^{-ikL}}{S}\right]\right)^2 = -\frac{R}{T}, \quad (A.7a) \\
\left(Re\left[\frac{e^{-ikL}}{S}\right] - \cos(\phi)\right)Im\left[\frac{e^{-ikL}}{S}\right] = 0. \quad (A.7b)
\end{align*}$$

From Eq.(A.7a) it follows that $Im\left[\frac{e^{-ikL}}{S}\right] \neq 0$. Therefore the Eq.(A.7b) leads to the following dispersion equation

$$Re\left[\frac{e^{-ikL}}{S}\right] = \cos(\phi). \quad (A.8)$$

Above dispersion equation is familiar from the persistent current problem.

Further we calculate the quantum mechanical current

$$I^{(l)} = -\frac{\hbar}{m_e} \text{Im} \left[\psi^{(l)} \frac{\partial \psi^{(l)*}}{\partial x} - \frac{e^2}{m_e} \frac{\Phi}{L} |\psi^{(l)}|^2\right], \quad (A.9)$$

carried by some eigenstate. Using the wave function Eq.(A.2) we get:

$$I^{(l)} = \pm \left( |A^{(l)}|^2 - |B^{(l)}|^2 \right). \quad (A.10)$$

Here $v^{(l)} = \hbar k^{(l)}/m_e$ is an electron velocity.

From the Eq.(A.5) we can express, say, $B^{(l)}$ in terms of $A^{(l)}$:

$$B^{(l)} = A^{(l)} \frac{S}{S_{22}} \left(\frac{e^{-ikL}}{S} - e^{i\phi}\right). \quad (A.11)$$

Substituting Eq.(A.11) in Eq.(A.9) we get:

$$I^{(l)} = I_0^{(l)} \sin(\phi),$$

$$I_0^{(l)} = -2ev^{(l)}|A^{(l)}|^2 \frac{T^{(l)}}{Re^{(l)}} \left(Im\left[\frac{e^{-ik^{(l)}L}}{S}\right] + \sin(\phi)\right). \quad (A.12)$$

The coefficient $A^{(l)}$ can be determined from the normalization condition:

$$|A^{(l)}|^2 + |B^{(l)}|^2 = L.$$
For instance, if $S = i t e^{i \chi}$ and $t \ll 1$ then

$$I_0 = -\frac{e_p(t)}{L} t \cos(k(l)L + \chi).$$

We can see, if the magnetic flux is not zero (or more precisely, is not either integer or half-integer of the magnetic flux quantum) then the eigenstates carry currents, i.e., they are running waves.

2. A ring with asymmetric scatterer

It is well known that using some gauge transformation the magnetic flux can be removed from the Schrödinger equation to the boundary conditions. In our case this means some transformation for the scattering matrix. From the equation (A.5) one can see what have to happen. If we put the magnetic flux (penetrating the ring) to be equal to zero and transform the transmission amplitudes as follows

$$S_{12} = S e^{-i \phi}, \quad S_{21} = S e^{i \phi},$$

then neither the dispersion equation (A.6) nor the circulating current Eq.(A.12) do not change.

The spectrum and the circulating current are the same in the following two cases:

$$\Phi \neq 0, \quad \hat{S} = \begin{pmatrix} S_{11} & S \\ S & S_{22} \end{pmatrix}$$

$$\Phi = 0, \quad \hat{S} = \begin{pmatrix} S_{11} & S e^{-i \phi} \\ S e^{i \phi} & S_{22} \end{pmatrix}$$

Therefore we can conclude that the asymmetric scatterer

$$S_{12} \neq S_{21},$$

induces a circulating current in a ring Fig.15b.

Note that the scatterer with such scattering properties can not push a current between the reservoirs, since the probabilities for transmission to the left and to the right are the same: $|S_{12}|^2 = |S_{21}|^2 \equiv |S|^2.$