NONLINEAR NORMAL VIBRATION MODES AND THEIR APPLICATIONS IN SOME APPLIED PROBLEMS

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Abstract

Nonlinear normal vibration modes (NNMs) are a generalization of the normal vibrations in the linear systems. In conception of NNMs by Kauderer-Rosenberg all position coordinates can be defined from any one of them. In conception of NNMs by Lyapunov-Shaw-Pierre all phase coordinates can be defined from two selected ones. Curvilinear trajectories of NNMs in a configuration space, or in a phase space, can be obtained as power series.

The NNMs theory is used to study vibrations of some linear structure connected with the single-DOF nonlinear absorber. An essentially nonlinear oscillator, a snap-through truss with three equilibrium positions, and a vibro-impact oscillator are chosen as absorbers. Construction and stability analysis of the NNMs are presented. If the localized mode is stable, and the non-localized vibration mode is unstable, the vibration energy is concentrated in the absorber.

Free damped oscillations of the double tracked road vehicle with a nonlinear response of the suspension can be considered by the NNMs theory too. The 7-DOF nonlinear model is used to analyze the suspension dynamics with smooth characteristics. The quarter-car model is considered for a case of the non-smooth characteristic of the shock absorber.

Key words

Nonlinear normal modes, absorber, suspension

1 Introduction

Nonlinear normal vibrations modes (NNMs) are a generalization of the normal vibrations of the linear systems. In the normal mode, a finite-dimensional system behaves like a single DOF conservative one, and all position coordinates can be analytically parametrized by any one of them. This conception of NNMs was proposed by Kauderer and Rosenberg.

Rosenberg [Rosenberg, 1962, 1966] defined NNMs as ‘vibrations in unison’ and introduced a broad class of essentially nonlinear conservative systems allowing for NNMs with rectilinear trajectories in a configuration space.

In general, the NNMs trajectories are curvilinear instead of straight lines in linear systems. The power series method was proposed in [Manevich and Mikhlin, 1972; Manevich, Mikhlin and Pilipchuk, 1989] to construct the curvilinear trajectories of NNMs. Shaw and Pierre reformulated the concept of NNMs for a general class of nonlinear discrete oscillators [Shaw and Pierre, 1991, 1993]. The analysis is based on the computation of invariant manifolds on which the NNM oscillations take place. All phase coordinates can be well defined from a pair of phase coordinates. This idea was first proposed by Lyapunov [Lyapunov, 1947]. In [Mikhlin, 1995] Padé approximations are used for an analysis of NNMs with large amplitudes.

Rauscher’s ideas and the power-series method for trajectories in a configurational space are used in the construction of NNMs in nonautonomous and self-excited systems, close to conservative ones [Mikhlin, 1974; Mikhlin and Morgunov, 2001]. In such systems some additional *potentiality conditions* must be used. It means that a loss of energy on the average over the period of the periodic solutions under consideration, is absent.

Basic results on NNMs are presented in the book by Vakakis et al. [Vakakis et al, 1996] which describes quantitative and qualitative analyses of NNMs, including localized modes, an analysis of stability, and an investigation of NNMs in distributed systems.

2 Nonlinear normal modes in a system with an essentially nonlinear absorber

Numerous publications contain an analysis of different devices for the vibration absorption of mechanical systems. The general theory of the linear/nonlinear absorbers is presented in [Kolovski, 1966; Frolov, 1995].

Oscillations of the two-DOF system which contains the essentially nonlinear absorber can be studied by the NNMs approach. Both the non-localized, and the localized vibration modes are possible here. The localized NNM is appropriate for the absorption, when the main linear system and the absorber have small and large amplitudes, respectively. If the localized mode is stable, and the non-localized vibration mode is unstable, the vibration energy is concentrated in the absorber. A system is considered where the nonlinear single-DOF absorber connected with a fixed point by cubically nonlinear spring (Fig.1). Here a mass of the absorber m is essentially smaller than that of the linear subsystem M .

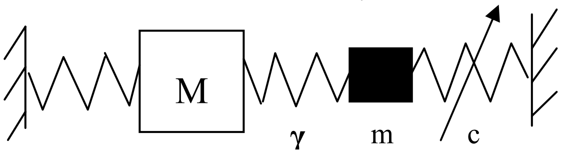


Fig.1. The system under consideration

One has the following equations of motion:

$$\begin{cases} \varepsilon m \ddot{x} + c x^3 + \gamma (x - y) = 0 \\ M \ddot{y} + \omega^2 y + \gamma (y - x) = 0 \end{cases} \quad (1)$$

Here x and y are the displacements of the absorber and main elastic systems respectively, ω^2, γ and c are stiffness coefficients of the springs, ε is the formal small parameter.

Eliminating t from the equations (1) by using the system energy integral, we can derive an equation to describing the NNM trajectory $y(x)$ [14]. The zero approximation with respect to ε ($\varepsilon = 0$) gives us the following: $y_0 = x + \frac{c}{\gamma} x^3$. This is the **non-localized vibration mode.** The classical procedure of the small parameter method permits to present the NNM trajectory as power series with respect to ε. The **localized vibration mode** can be analized if the next time transformation is introduced: $t = \sqrt{\varepsilon}\tau$. One has in this case the following zero approximation with respect to ε: $y_0 = 0$. The first approximation solution can be obtained in the form of power series by x. Test numerical calculations show a good accuracy of the analytical solutions.

A harmonic approximation of the non-localized vibration mode permits to reduce a problem of the NNM stability to a single variational equation in the form of the standard Mathieu equation parameters $\varepsilon^{\bullet}, \delta^{\bullet}$ (these parameters are defined by the system (1) parameters). The vibration amplitudes are not limited here. The Fig. 2 shows a position of point, corresponding to the NNM, in a plane of the Mathieu equation parameters (for some fixed values of the parameter γ and some other parameters). Direction of the vibration amplitudes increase is showed by the arrows. The non-localized vibration mode is situated in the region of instability (this region is shaded), but for not very big vibration amplitudes the solution is near the boundary of the region. A problem of the localized vibration mode stability can be reduced to the Mathieu equation too. It be obtained that the localized mode is stable for almost all values of the system parameters [Mikhlin and Reshetnikova, 2005].

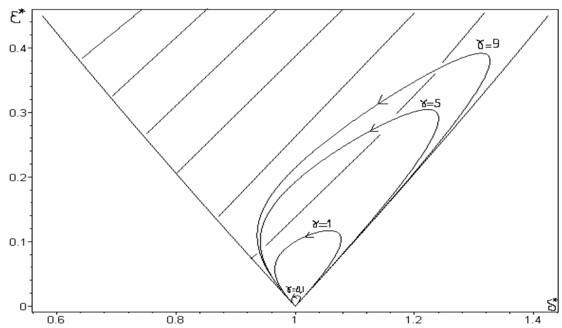


Fig. 2: Region of the non-localized mode instability.

Other approach of the NNMs stability analysis is based on the so-called algebraization by Ince. In this case a new independent variable associated with the solution under consideration is chosen [Ince, 1926]. Then the variational equation is converted to the equation with singular points. This approach was used earlier in the NNMs theory [Mikhlin and Zhupiev, 1997]. This algebraization gives us more exact results than those obtained by the reduction of the stability problem to the Mathieu equation [Mikhlin and Reshetnikova, 2005]. One introduces the following transformation of the independent variable: $t \to x$. One has after some transformations the variational equation with singular points. It can be demonstrated that in this case solutions corresponding to boundaries of the stability/instability regions are the following: $v_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, $v_2 = \sqrt{X_0{}^2 - x^2} \cdot \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots\right)$.

Substituting the series to the variational equation and matching respective powers of x, we can write recurrent systems of linear algebraic equations to determine coefficients of the expansions. These systems have nontrivial solutions if their determinants are equal to zero. One obtains from here the stability/instability regions boundaries. It generally corresponds to results obtained for the Mathieu equation. But for small vibration amplitude the non-localized vibration mode is situated in the domain of stability near its boundary and gets into the instability domain if the vibration amplitude increases. Additional narrow instability regions are obtained for the localized vibration mode in a case when the stiffness coefficient γ is small.

3 Snap-through truss as a vibration absorber

Possibility of the elastic oscillations absorption by means of the snap-through truss is considered here. The nonlinear absorber with three equilibrium positions (snap-through truss) is attached to the linear oscillator. By assumption, the truss is shallow and its mass and stiffness are significantly smaller then the corresponding parameters of the linear system.

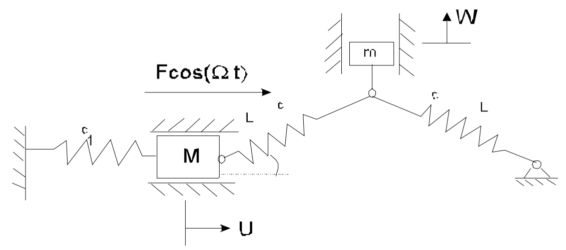


Fig.3. The system with the snap-through truss

Let (U, W) be the generalized coordinates; L is a length of the spring; φ is the angle, corresponding to the equilibrium position; κ is a spring stiffness of the truss; κ_1 is a stiffness of the main elastic system. The dimensionless variables $u = U/L; w = W/L$ and $t = M\tau/k_1$ are introduced. Let us introduce, $\mu = \varepsilon\bar{\mu}; \gamma = \varepsilon\bar{\gamma}; \varepsilon \prec\prec 1$. The new variable is introduced too: $u_1 = u + \gamma(1-k)/(1+\gamma)$. Retaining the linear, quadratic and cubic terms by u_1 and w we can write the equations of motion as

$$\ddot{u}_1 + (1+\varepsilon\bar{\gamma})u_1 - \frac{\varepsilon\bar{\gamma}}{\rho^3}u_1 w^2 - \frac{\varepsilon\bar{\gamma}}{2\rho^2}w^2 = 0;$$
$$\bar{\mu}\ddot{w} - \bar{\gamma}\alpha^2 w - \frac{\bar{\gamma}}{\rho^2}wu_1 + \frac{\bar{\gamma}\beta^2}{2}w^3 = 0, \qquad (2)$$

where

$$\rho = \frac{\gamma+\kappa}{1+\gamma}; \alpha^2 = \frac{1}{\rho} + \frac{1}{\kappa} - 2; \beta^2 = \frac{1}{\rho^3} + \frac{1}{\kappa^3}.$$

To study the periodic motions of the system (2) with large amplitudes we determine here the NNM in the form $u_1 = u_1(w)$ [Avramov and Mikhlin, 2004]. The solution is presented as power series, which is then substituted into the corresponding equation and to the boundary conditions to obtain the coefficients of the series. Numerical calculations show a good accuracy of the analytical solution. The snap-through truss has significant amplitudes of oscillations and the linear oscillator has small amplitudes. If such motions are stable, the vibration absorption is guaranteed.

4 The snap-through periodic motion stability

A small curvature of the obtained NNMs is used here to analyze their stability [Avramov and Mikhlin, 2004]. Let us introduce new variables (ξ,η). The ξ axis is directed along the rectilinear approximation of the NNM trajectory and the orthogonal variation η(t) defines the orbital stability of NNMs. One considers a stability of the periodic motions with large amplitudes taking into account that $u_1 = O(\varepsilon)$ and introducing the following representation: $w = w_0 + O(\varepsilon)$. We study variations η(t) of periodic motions $\bar{u}_1$: $u_1 = \bar{u}_1 + \eta$. One obtains after some transformations the following variational equation:

$$\ddot{\eta} + (1 + \varepsilon\bar{\gamma} - \frac{\varepsilon\bar{\gamma}}{\rho^3}w_0^2)\eta = 0, \qquad (3)$$

where $w_0 = \sqrt{2}\frac{\alpha}{\beta}\sqrt{1+\sqrt{1+4H}}\,\mathrm{cn}(p\alpha\tau\sqrt[4]{1+4H};k)$.

Here k is the elliptic integral modulus; H is the oscillator total energy. Substituting the Fourier-series expansion of $\mathrm{cn}^2(t,k_0)$ into (3), we can rewrite it as the Hill's equation which is analyzed by the multiple scales method. Resonances of the orders s (s=1,2,3…) are considered: $s\Omega_* = 2\omega_0 + \varepsilon\sigma_s$, where σ_s is the detuning parameter. Since the snap-through truss is shallow, the next notations can be introduced: $1 - c = \varepsilon_* c_1; \varepsilon_* \prec\prec 1$. After some analysis of the corresponding modulation equations, which are not presented here, we can obtain the stable/unstable region boundaries on the system parametric plane (c, W_*) as

$$\frac{\sqrt{2}cK(k_0)}{p\pi s} = \frac{W_*}{\sqrt{2c}}\left(1 - \varepsilon_*\frac{c^2 c_1}{W_*^2}\right) + O(\varepsilon_*^2) + O(\varepsilon), \qquad (4)$$

where $k_0^2 = \frac{1}{2} + \varepsilon_*\frac{c^2 c_1}{W_*^2} + O(\varepsilon_*^2)$; $W_* = W_{max}$;

$$K(k_0) = K\left(\frac{1}{\sqrt{2}}\right) + \varepsilon_* K'\left(\frac{1}{\sqrt{2}}\right)\frac{c^2 c_1}{\sqrt{2}W_*^2} + O(\varepsilon_*^2).$$

Here $K(k_0)$ is the complete elliptic integral of the first kind. Fig. 4 shows curves (OA_1),(OA_2),(OA_3) in the plane (c, W_*), which are in correspondence with the equation (4). The boundaries of the stable/unstable regions are shown qualitatively as curves $(B_1C_1D_1)$, $(B_2C_2D_2)$, $(B_3C_3D_3)$. If φ is small, the unstable oscillations regions have the order O(ε). If value φ is increased, the width and number of the unstable regions are decreased. We can choose such values of φ, that is the periodic motions are always stable.

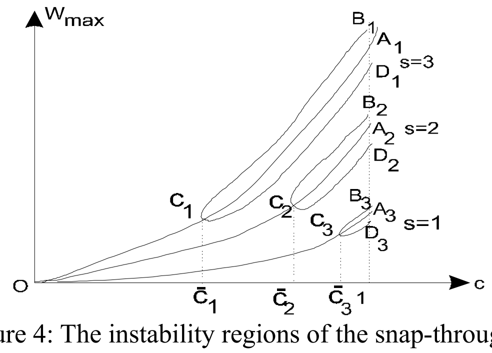


Figure 4: The instability regions of the snap-through motion.

Remark

Note that a similar analysis of localized and non-localized NNMs is realized for the two-DOF system containing a vibro-impact oscillator as absorber. Besides, the forced resonances are investigated in such systems with three types of absorbers, namely, an essentially nonlinear oscillator with a single equilibrium position [Mikhlin and Reshetnikova, 2005]; a snap-through truss with three equilibrium positions [Avramov and Mikhlin, 2004]; and a vibro-impact oscillator.

5 Principal model of the vehicle suspension nonlinear dynamics.

7-DOF model of the vertical and axial vehicle dynamics is considered here for a case of independent-solid axle suspension to predict the vehicle body and wheel states. It is possible to consider, by using this model, all principal vehicle motions. The correctness of this model is substantiated by comparison with some experiments [Wong, 1993; Hyo-Jun Kim, Hyun Seok Yang, Young-Pil Park, 2002; Pilipchuk et al., 2006]. It is known that nonlinear effects in the suspension dynamics are important if corresponding displacements are in the order of 0.05-0.1 m, or larger. Here the smooth nonlinear spring characteristics in front and rear suspensions are taken into account. NNMs and corresponding amplitude-frequency relations are obtained for the nonlinear 7-DOF system.

In order to describe the double-tracked road vehicle dynamics, the 7-DOF mathematical model is used (Fig.5). It considers the heave, roll and pitch motions of the car body. Here z is the vertical displacement, α is the pitch angle, β is the roll angle, x_i is the vertical displacement of i-th suspended mass which are equivalent to the wheel, d_1, d_2 are the front and the rear track widths and l_1, l_2 are the front and the rear wheel bases. In this model tires are presented as elastic elements with linear characteristics. The suspension is characterized by nonlinear elastic characteristics of the front and rear springs, and linear damping characteristics. Typical elastic characteristics are shown in the Fig. 6.

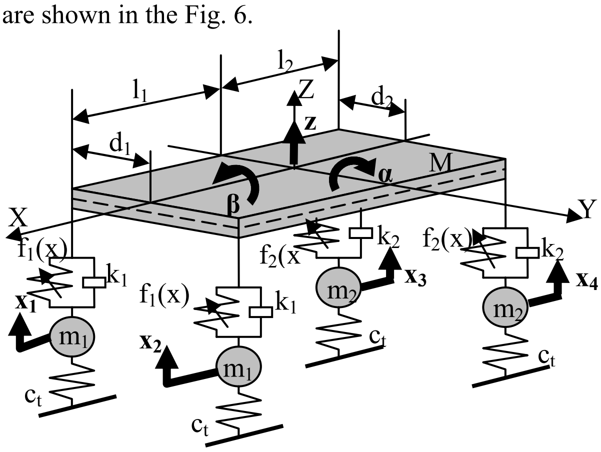


Fig. 5. Mathematical model of a double-tracked road vehicle under consideration

One has seven generalized coordinates to describe vibrations of the model.

6 Nonlinear normal modes and the transients (smooth characteristics of suspension)

To obtain NNMs one chooses a couple of new independent variables (u, v), where u is one of the

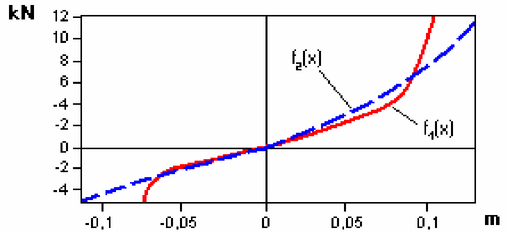
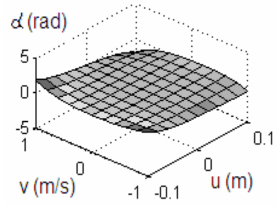


Fig. 6. Nonlinear characteristics of the front $f_1(x)$

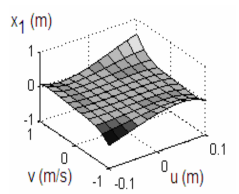
generalized coordinates, and v is the corresponding generalized velocity. According to the Lyapunov-Shaw-Pierre approach, the nonlinear normal mode is such regime when all generalized coordinates and velocities are univalent functions of the selected couple of variables. One presents the NNMs in the form of the power series by independent variables u and v:

$$x_i = X_i(u,v) = a_{1i}u + a_{2i}v + a_{3i}u^2 + a_{4i}uv + a_{5i}v^2 + \ldots,$$
$$y_i = Y_i(u,v) = b_{1i}u + b_{2i}v + b_{3i}u^2 + b_{4i}uv + b_{5i}v^2 + \ldots. \quad (5)$$

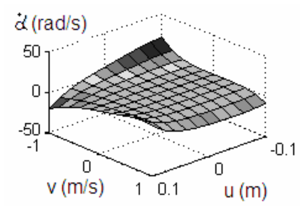
To calculate coefficients of the series (5) one has a system of algebraic equations. One of the NNMs obtained here is shown in the Fig. 7, where the coordinate z is chosen as the independent variable u. If the NNM in the form (5) is obtained, the series are substituted to equations of motion, and functions $u = u(t)$ and $v = v(t)$ can be obtained too. As a result, seven NNMs were determined. Numerical calculations show a good accuracy of the obtained analytical results. To construct the NNMs skeletons the harmonic linearization method together with a continuation procedure were used.

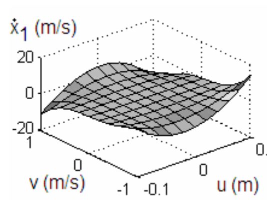


a) The angle α versus u and v



b) The displacement x_1 versus u and v





c) The angular velocity $\dot{\alpha}$ versus u and v

d) The velocity $\dot{x}_1$ versus u and v

Fig. 7. The first NNM (independent variable z)

In a case of large initial displacements only the low-frequency vibration modes of the car body are stable, and other vibration modes tend to the low-frequency modes.

7. Nonlinear normal modes and the transients (shock absorber, non-smooth characteristic)

To investigate the suspension dynamics taking into account a non-smooth characteristic of the shock absorber, the quarter-car model is considered (Fig.8).

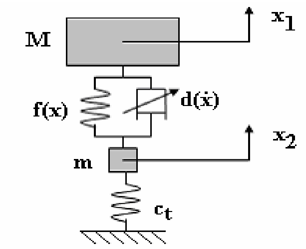


Fig. 8. The quarter-car model

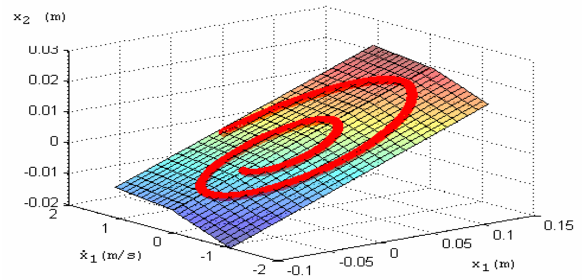
Equations of motion for the quarter-car model are

$$\begin{cases} M\ddot{x}_1 + f(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) = 0, \\ m\ddot{x}_2 + f(x_2 - x_1) + d(\dot{x}_2 - \dot{x}_1) + c_t x_2 = 0. \end{cases} \quad (6)$$

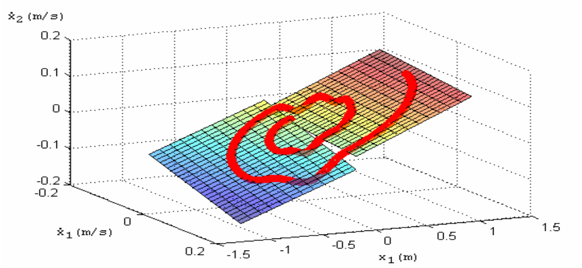
where $f(x)$ is a stiffness function, $d(\dot{x})$ is a piecewise damping function of the suspension, namely,

$$d(\dot{x}) = \begin{cases} d_1(\dot{x}), & \dot{x}_1 - \dot{x}_2 < 0, \\ d_2(\dot{x}), & \dot{x}_1 - \dot{x}_2 \geq 0. \end{cases} \quad (7)$$

An analysis of NNMs is made for a case of non-symmetric piecewise linear damping characteristic. Both NNMs were obtained by using the method which is described below. Motions on places, corresponding to NNMs, are shown in Fig. 9. After each gap of the piecewise linear damping characteristic, the short transient from one place to another one can be observed.



a) The displacement x_2 versus u and v



b) The velocity $\dot{x}_2$ versus u and v

Fig. 9. First NNM. Independent variables x_1 and the corresponding velocity (piecewise linear system)

More realistic damping characteristic is presented in Fig. 10.

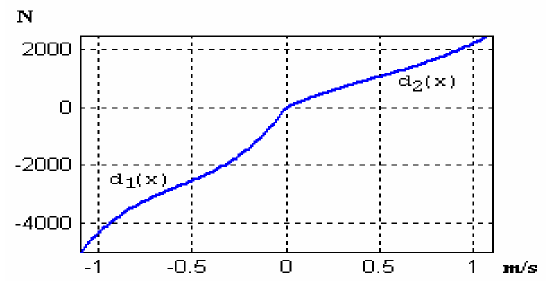
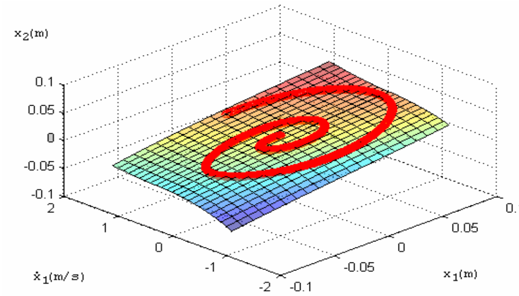
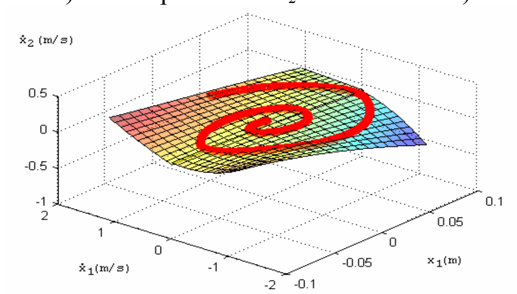


Fig. 10. The piecewise cubic shock absorption characteristic in the suspension

Corresponding NNMs obtained in this case and transients from one surface to another one after gap (or "switching") in the piecewise cubic damping characteristic are shown in Figs. 11 and 12.

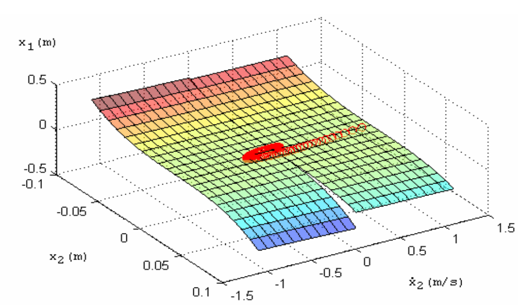


a) The displacement x_2 versus u and v)

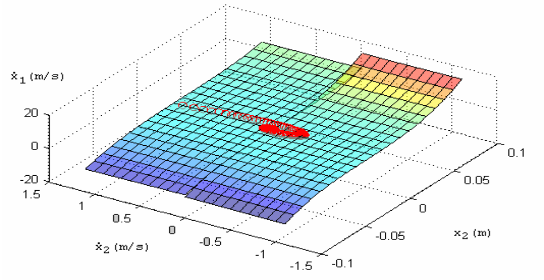


b) The velocity $\dot{x}_2$ versus u and v

Fig. 11. First NNM. Independent variables x_1 and the corresponding velocity (piecewise cubic system)



a) The displacement x_1 versus u and v



b) The velocity $\dot{x}_2$ versus u and v

Fig. 12. Second NNM. Independent variables x_2 and the corresponding velocity (piecewise cubic system)

8 Conclusion

At the present time, it is known that NNMs are typical periodical solutions in *n*-DOF nonlinear conservative systems. Moreover, normal or quasinormal vibrations exist in broad classes of nonlinear, near-conservative systems.

It is shown that methods of the NNMs theory permit to describe a behavior of systems containing nonlinear dynamical absorbers. The nonlinear dynamics of the double tracked road vehicle with a nonlinear response of the suspension can be analyzed by using the NNMs approach both for a case of the smooth nonlinear characteristic of suspension, and for a case of the non-smooth characteristic of the shock absorber.

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