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Study of Two-Dimensional Axisymmetric Breathers Using Padé Approximants

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Abstract. We analyze axisymmetric, spatially localized standing wave solutions with periodic time dependence (breathers) of a nonlinear partial differential equation. This equation is derived in the 'continuum approximation' of the equations of motion governing the anti-phase vibrations of a two-dimensional array of weakly coupled nonlinear oscillators. Following an asymptotic analysis, the leading order approximation of the spatial distribution of the breather is shown to be governed by a two-dimensional nonlinear Schrödinger (NLS) equation with cubic nonlinearities. The homoclinic orbit of the NLS equation is analytically approximated by constructing $[2N \times 2N]$ Padé approximants, expressing the Padé coefficients in terms of an initial amplitude condition, and imposing a necessary and sufficient condition to ensure decay of the Padé approximations as the independent variable (radius) tends to infinity. In addition, a convergence study is performed to eliminate 'spurious' solutions of the problem. Computation of this homoclinic orbit enables the analytic approximation of the breather solution.

Key words: Nonlinear localization, nonlinear differential equations, Padé approximations.

1. Introduction

Standing and traveling waves in one dimensional lattices [1–4] and in systems described by nonlinear partial differential equations [5–8] have been extensively studied in the literature. In [9] a numerical technique based on geometrical arguments in phase space was developed to study axisymmetric standing waves of the nonlinear Schrödinger (NLS) equation; the existence and properties of standing wave solutions in NLS were studied in [10–12]. MacKay and Aubry [13] proved the existence of weakly and strongly localized breathers (i.e., time-periodic and spatially localized waves considered in appropriate co-ordinate systems) in Hamiltonian systems consisting of weakly coupled nonlinear oscillators; in their work they used concepts from analytic continuation of solutions and functional analysis. In the work by Akylas [14] three-dimensional effects on soliton and periodic wave interactions and on water wave propagation are reviewed.

In this work we analyze axisymmetric standing breathers of a nonlinear partial differential equation with two independent variables. As in [8] the problem formulation is performed by regarding the standing breathers as localized nonlinear normal modes (NNMs) [15], and deriving the nonlinear ordinary differential equations that govern the leading orders approximations to the breather envelopes. Then, we use diagonal Padé approximants [16] to develop analytic approximations for the homoclinic orbits of these leading order differential equations; these solutions are shown to provide approximations for the localized spatial distributions of the breather envelopes. Although the analysis is carried out for a specific nonlinear partial differential equation, the methodology is sufficiently general to be applicable to general classes of partial differential equations with two or more independent variables that admit standing breather-type solutions. Moreover, the method of Padé approximants developed in this work can be used to compute analytical approximations of homoclinic and heteroclinic orbits of nonlinear dynamical systems with phase spaces of dimensions greater or equal to two.

2. Asymptotic Analysis

Consider the small transverse oscillations of a two-dimensional chain of rigid particles unbounded in the plane. Each particle has transverse diplacement $v_{m,n}(t)$, is grounded by a nonlinear stiffness with linear and cubic characteristics, and is coupled to its neighboring particles by massless strings. Assuming *anti-phase* motions between any two neighboring particles, the transverse vibrations of the particles can be approximately modeled by the following nonlinear partial differential equation:

$$\begin{aligned} &u_{tt} + \alpha(u + \varepsilon\mu u^3) + \varepsilon\lambda(u_{xx} + u_{yy}) = 0, \quad 0 < \varepsilon \ll 1, \quad \alpha, \mu, \lambda > 0 \\ &\quad -\infty < x < +\infty, \quad -\infty < y < +\infty, \quad t \geq 0, \end{aligned} \tag{1}$$

where the short-hand notation for partial differentiation was adopted, e.g., $(\cdot)_{xy} \equiv \partial^2(\cdot)/\partial x\partial y$. Equation (1) is obtained as the 'continuum approximation' [1, 17] of the bi-infinite set of ordinary differential equations governing the transverse vibrations of the two-dimensional discrete array after the transformation of variables $u_{m,n}(t) = (-1)^n\ (-1)^m v_{m,n}(t)$ has been imposed. In the continuum approximation, only leading-order discreteness effects are taken into account, and the discrete positional variables of the oscillators are transformed to a continuous variable with two spatial and one temporal independent variables. Higher order discreteness effects can be taken into account [18] but this is not performed here.

We seek axisymmetric standing breather solutions of this equation satisfying the following relations:

$$\lim_{|x^2+y^2|\to\infty} u(x,y,t) = 0, \quad u(x,y,t+T) = u(x,y,t), \tag{2}$$

where T denotes the period of the standing wave oscillation. To compute these solutions we extend the methodology first developed in [8] where the one-dimensional analog of Equation (1) was studied. An added complication in the two-dimensional case is radial dispersion which, as shown below, introduces a dispersion/dissipative term in the equation governing the leading order approximation of the spatial distribution of the solution. As in [8] we consider a 'reference position' (x_0, y_0) and a 'reference displacement' $u(x_0, y_0, t) \equiv u_0(t)$. On a breather solution of (1), $u(x, y, t)$ can be parametrized in terms of the reference displacement as follows:

$$u(x,y,t) = U[x,y,u_0(t)], \tag{3}$$

where $U[x, y, u_0(t)]$ is referred to as the *modal function*. Once the modal function and the reference displacement are computed, the response of the system is defined by (3).

We now formulate a well-posed problem in terms of $U[x, y, u_0(t)]$ that can be solved by asymptotic analysis. Considering the first integral of motion for (1) (where E represents the value of the total energy),

$$E = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [u_t^2 + \alpha u^2 + (\varepsilon/2)\alpha\mu u^4 - \varepsilon\lambda(u_x^2 + u_y^2)] \, \mathrm{d}x \, \mathrm{d}y \tag{4a}$$

expressing $u(x, y, t)$ by (3), and using the chain rule of differentiation, we obtain the following expression for the square of the derivative of the reference displacement:

$$\left(\frac{\mathrm{d}u_0}{\mathrm{d}t}\right)^2 = \frac{2E - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\alpha U^2 + (\varepsilon/2)\alpha\mu U^4 - \varepsilon\lambda(U_x^2 + U_y^2)] \, \mathrm{d}x \, \mathrm{d}y}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (U_{u_0})^2 \, \mathrm{d}x \, \mathrm{d}y}. \tag{4b}$$

It is assumed that $E < \infty$ for the types of motions considered herein. The acceleration of the reference point is computed by evaluating the equation of motion (1) at the reference point:

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}t^2} = [-\alpha(U + \varepsilon\mu U^3) - \varepsilon\lambda(U_{xx} + U_{yy})]_{(x,y)=(x_0,y_0)}. \tag{5}$$

Relations (4b) and (5) are now used to express the governing equation of motion in terms of the modal function and the reference displacement by means of (3) and the chain rule of differentiation:

$$\begin{aligned} &\left\{ \frac{2E - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\alpha U^2 + (\varepsilon/2)\alpha\mu U^4 - \varepsilon\lambda(U_x^2 + U_y^2)] \, \mathrm{d}x \, \mathrm{d}y}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (U_{u_0})^2 \, \mathrm{d}x \, \mathrm{d}y} \right\} U_{u_0 u_0} \\ &\quad + \{-\alpha[U(x_0, y_0, u_0) + \varepsilon\mu U^3(x_0, y_0, u_0)] - \varepsilon\lambda[U_{xx}(x_0, y_0, u_0) + U_{yy}(x_0, y_0, u_0)]\} U_{u_0} \\ &\quad = -\alpha U - \varepsilon\alpha\mu U^3 - \varepsilon\lambda(U_{xx} + U_{yy}). \end{aligned} \tag{6a}$$

Equation (6a) must be solved subject to the following conditions which ensure (i) that relations (2) are satisfied [Equation (6b)]; (ii)‘that the definition (3) is compatible with the definition of the reference displacement [Equation (6c)]; and (iii) that the asymptotic solution of (6a) is analytically extended up to the point of maximum potential energy [8] [Equation (6d)]:

$$\lim_{|x^2+y^2| \to \infty} U[x, y, u_0(t)] = 0, \quad u_0(t + T) = u_0(t), \tag{6b}$$

$$U[x_0, y_0, u_0(t)] = u_0(t), \tag{6c}$$

$$\begin{aligned} &+\{-\alpha[U(x_0, y_0, u_0^*) + \varepsilon\mu U^3(x_0, y_0, u_0^*)] \\ &\quad - \varepsilon\lambda[U_{xx}(x_0, y_0, u_0^*) + U_{yy}(x_0, y_0, u_0^*)]\} U_{u_0}(x, y, u_0^*) \\ &\quad = -\alpha U(x, y, u_0^*) - \varepsilon\alpha\mu U^3(x, y, u_0^*) - \varepsilon\lambda[U_{xx}(x, y, u_0^*) + U_{yy}(x, y, u_0^*)], \end{aligned} \tag{6d}$$

where u_0^* denotes the maximum value attained by $u_0(t)$ when the system reaches its maximum potential energy value; u_0^* is computed in terms of the total energy E by the following relation:

$$\begin{aligned} 2E = &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \{\alpha U^2(x, y, u_0^*) + (\varepsilon/2)\alpha\mu U^4(x, y, u_0^*) \\ &\quad - \varepsilon\lambda[U_x^2(x, y, u_0^*) + U_y^2(x, y, u_0^*)]\} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{7}$$

Considering relations (6a) and (7), it becomes clear why the condition of analytic continuation (6d) must be imposed: The points $u_0(t) = \pm u_0^*$ are regular singular points of (6a) (since the coefficient of the highest order derivative vanishes there), and condition (6d) ensures that asymptotic approximations developed in open intervals $|u_0(t)| < u_0^*$ are analytically extended up to the singular points. Moreover, since the system under consideration possesses cubic-type nonlinearities, the solutions of (6a–d) are odd with respect to the argument $u_0(t)$, and, the problem has to be solved only in the half-interval $0 \leq u_0(t) \leq u_0^*$.

Following a methodology similar to [8], the solution of (6a–d) is expressed in the following series form:

$$U[x, y, u_0(t)] = \sum_{k=0}^{\infty} \varepsilon^k U^{(k)}[x, y, u_0(t)], \tag{8a}$$

where the leading order approximation is expressed as,

$$U^{(0)}[x, y, u_0(t)] = a_1^{(0)}\ (x, y) u_0(t) \tag{8b}$$

in view of the separation of space and time in (1) for $\varepsilon = 0$. Higher-order approximations are not, in general, separable in space and time and are expanded in series as follows:

$$U^{(k)}[x, y, u_0(t)] = \sum_{m=1}^{\infty} a_m^{(k)}\ (x, y) u_0^m(t), \quad k \geq 1. \tag{8c}$$

Moreover, due to the compatibility relation (6c), the spatial coefficients in (8b, c) satisfy the relations:

$$a_1^{(0)}(x_0, y_0) = 1, \quad a_m^{(k)}\ (x_0, y_0) = 0, \quad m = 1, 2, \ldots, k \geq 1. \tag{8d}$$

Since in the following asymptotic analysis the series (8a) and (8c) will need to be truncated, it is necessary to remark that the resulting expresions will be valid only for ε and $u_0(t)$ sufficiently small. Hence, we will be computing the solutions of (6a–d) in small neighborhoods of the origin of the parameter plane $[\varepsilon, u_0(t)]$. We now consider each order of approximation separately.

$O(1)$ terms

The equation governing the O(1) approximation to the solution is given by,

$$\begin{aligned} &U^{(0)}[x_0, y_0, u_0(t)] U_{u_0}^{(0)}[x, y, u_0(t)] = U^{(0)}[x, y, u_0(t)] \\ &\quad \Rightarrow a_1^{(0)}\ (x, y)[1 - a_1^{(0)}\ (x_0, y_0)] u_0(t) = 0, \end{aligned} \tag{9}$$

where the separation of variables (8b) was imposed. For nontrivial solutions, we require that $a_1^{(0)}\ (x_0, y_0) = 1$, which is identical to the first of the compatibility relations (8d). Hence, the balancing of O(1) terms in (6a–d) does not provide any new information for the solution and higher order terms must be considered.

$O(\varepsilon)$ terms

Substituting (8a) into (6a–d) and matching terms of $O(\varepsilon)$ we obtain the equations governing $U^{(1)}[x, y, u_0(t)]$. Expressing $U^{(1)}[x, y, u_0(t)]$ by the series expression (8c) with $k = 1$, and matching the coefficients of respective powers of $u_0(t)$, we obtain the following equations governing the spatial distributions $a_1^{(0)}\ (x,y)$ and $a_m^{(1)}\ (x,y)$:

$$\nabla^2 a_1^{(0)}\ (x,y) - \left[\nabla^2 a_1^{(0)}\ (x_0,y_0) + \frac{3\alpha u_0^{*2}}{4\lambda}\right] a_1^{(0)}\ (x,y) + \frac{3\alpha u_0^{*2}}{4\lambda}\ a_1^{(0)^3}\ (x,y) = 0,$$

$$a_3^{(1)}\ (x,y) = \frac{\lambda}{6\alpha u_0^{*2}}\ [\nabla^2 a_1^{(0)}(x_0,y_0) a_1^{(0)}\ (x,y) - \nabla^2 a_1^{(0)}\ (x,y)],$$

$$a_4^{(1)}\ (x,y) = 0,$$

$$\dots \tag{10}$$

where only terms up to $O[u_0^3(t)]$ were included. Complementing these relations are the compatibility Equations (8d) and the following set of limiting expressions:

$$\lim_{|(x-x_0)^2+(y-y_0)^2|\to\infty} a_1^{(0)}\ (x,y)\ =\ 0,$$

$$\lim_{|(x-x_0)^2+(y-y_0)^2|\to\infty} a_m^{(1)}\ (x,y)\ =\ 0, \qquad m = 1,2,\dots \tag{11}$$

The approximation $a_1^{(1)}\ (x,y)$ is computed at the next order of approximation.

The first of Equations (10) governs the $O(1)$ leading approximation for the envelope of the breather. Introducing polar instead of Cartesian co-ordinates $x - x_0 = r\sin\theta, y - y_0 = r\cos\theta$, the Laplacian operator in the equation for $a_1^{(0)}$ is expressed as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\ \frac{\partial}{\partial r} + \frac{1}{r^2}\ \frac{\partial^2}{\partial \theta^2}\ .$$

Since we seek axisymmetric breather solutions we assume that there is no θ-dependence in the solution, $a_1^{(0)} = a_1^{(0)}(r)$. Furthermore, we introduce the following transformation of variables,

$$\kappa = \nabla^2 a_1^{(0)}(0), \quad \beta = \frac{3\alpha\mu u_0^{*2}}{4\lambda}\ , \quad \zeta = (\kappa+\beta)^{1/2} r, \quad a_1^{(0)} = \mp\left(\frac{\kappa+\beta}{\beta}\right)^{1/2} \varphi(\zeta). \tag{12}$$

Using (12), and considering (8b), (10) and (11), the problem governing the leading order approximation $a_1^{(0)}$ assumes the nondimensional form,

$$\varphi''(\zeta) + \frac{1}{\zeta}\ \varphi'(\zeta) - \varphi(\zeta) + \varphi^3(\zeta) = 0, \quad \varphi(0) = \pm\left(\frac{\beta}{\beta+\kappa}\right)^{1/2}, \quad \lim_{\zeta\to\infty} \varphi(\zeta) = 0. \tag{13}$$

with an additional condition for the slope $\varphi'(0) = 0$, being dictated from symmetry arguments; moreover, the problem is solved in the domain $\zeta \geq 0$.

The solution of (13) provides the $O(1)$ spatial coefficient $a_1^{(0)}$ and the $O(\varepsilon)$ cubic spatial coefficient $a_3^{(1)}$ (through the third or relations (10)). The $O(\varepsilon)$ linear spatial coefficient $a_1^{(1)}$ cannot be determined at this order of approximation, and $O(\varepsilon^2)$ terms must be considered. In the following section we develop analytic approximations of (13) by means of Padé approximants.

Before proceeding with the asymptotic evaluation of $a_1^{(0)}$, we briefly comment on the computation of the reference displacement $u_0(t)$. From the previous derivations, the modal function for the breather is approximated as,

$$U[x, y, u_0(t)] = [a_1^{(0)}\ (x, y) + \varepsilon a_1^{(1)}\ (x, y)]u_0(t) + \varepsilon a_3^{(1)}\ (x, y)u_0^3(t) + O[\varepsilon u_0^5(t), \varepsilon^2], \tag{14}$$

where Cartesian co-ordinates are used. An equation for the determination of $u_0(t)$ is derived by substituting (14) into the governing equation of motion (1) and evaluating the resulting expression at the reference position (x_0, y_0):

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}t^2} + \alpha(1 + \varepsilon\lambda h_1)u_0 + \varepsilon\mu u_0^3 + O(\varepsilon^2) = 0, \tag{15a}$$

$$h_1 = \left\{ \frac{\partial^2 a_1^{(0)}}{\partial x^2} + \frac{\partial^2 a_1^{(0)}}{\partial y^2} \right\}_{(x,y)=(x_0,y_0)}. \tag{15b}$$

This equation is the classical Duffing oscillator subject to initial conditions $(u_0(0), \mathrm{d}u_0(0)/\mathrm{d}t) = (u_0^*, 0)$, and its exact solution can be expressed in terms of elliptic functions and integrals [19].

3. Padé Approximations

Problem (13) can be recognized as identical to the one governing the radially symmetric standing wave solutions of the nonlinear Schrödinger (NLS) equation in two dimensions. As discussed in [9] and other works, this problem possesses solutions that decay to zero as ζ tends to infinity and possessing arbitrary numbers of zeros. In [23] an existence theorem is given regarding the solutions of (13); it is proven that this problem possesses a discrete infinite spectrum (i.e., a countable infinity) of initial conditions $\varphi^{(i)}(0)$, ordered in the sequence $0 < \varphi^{(0)}(0) < \varphi^{(1)}(0) < \ldots$, and with the i-th solution corresponding to a localized response with i zeros (nodes). A numerical technique for computing these decaying solutions is given in [9], based on locating basin boundaries between attracting invariant curves in the three-dimensional phase space of the system. Additional numerical techniques for solving (13) are discussed in [24, 25].

In this section we develop a new technique for *analytically approximating* the decaying solution of (13) with initial condition $\varphi^{(0)}(0)$, corresponding to a localized breather with no nodes. The technique is based on diagonal Padé approximants [16], and can be similarly applied to study breather solutions with initial conditions $\varphi^{(i)}(0), i \geq 1$. As shown in [20, 21], Padé approximations can be used to determine the radius of convergence of asymptotic expansions of nonlinear oscillatory problems. This can be performed by studying the convergence of the poles and zeros of successive Padé approximants; based on this analysis, a transformation of variables can be introduced that eliminates the singularities from the perturbation expansions, leading to infinite radius of convergence. In an additional work [22] applications of Padé approximants in perturbation problems are discussed.

We begin our analysis by recognizing that the problem of computing the decaying solutions of (13) is identical to the problem of computing homoclinic orbits in the three-dimensional phase space of the nonlinear oscillator,

$$\varphi''(\zeta) + \frac{1}{\zeta}\,\varphi'(\zeta) - \varphi(\zeta) + \varphi^3(\zeta) = 0 \tag{16}$$

or equivalently, of computing the initial conditions $(\varphi(0), \varphi'(0)) = (\Phi, 0)$ for these orbits. In view of the previously introduced polar transformation and the anticipated symmetry of the solution, we restrict the analysis to $\zeta \geq 0$ and require that $\varphi(\zeta) = \varphi(-\zeta)$. Since the sought solutions are expected to be analytic functions of ζ, they can be expressed in Taylor series about $\zeta = 0$,

$$\varphi(\zeta) = \sum_{p=0}^{\infty} C_{2p}\zeta^{2p}, \tag{17a}$$

where the leading coefficients of the series were computed in terms of the (yet undetermined) initial displacement Φ using *Mathematica*:

$$\begin{aligned}
C_0 &= \Phi, \quad C_2 = \frac{1}{4}\,\Phi(1-\Phi^2), \quad C_4 = \frac{1}{64}\,\Phi(1-\Phi^2)(1-3\Phi^2),\\
C_6 &= \frac{1}{2304}\,\Phi(1-\Phi^2)(1-3\Phi^2) - \frac{3}{16}\,\Phi^3(1-\Phi^2)^2,\\
C_8 &= \frac{1}{64}\,[C_6(1-3\Phi^2) - 6\Phi C_2 C_4 - C_2^3],\\
C_{10} &= -\frac{1}{100}\,[(3\Phi^2-1)C_8 + 6\Phi C_2 C_6 + 3\Phi C_4^2 + 3C_4 C_2^2],\\
C_{12} &= -\frac{1}{144}\,[3\Phi^2 C_{10} + 6\Phi C_2 C_8 + 6\Phi C_4 C_6 + 3C_6 C_2^2 + 3C_2 C_4^2 - C_{10}],\\
&\ldots
\end{aligned} \tag{17b}$$

We now construct the $[2N \times 2N]$ diagonal Padé approximants corresponding to the truncated Taylor series (17a) of degree $4N$. In essence, the Padé approximants are Laurent series that converge to the Taylor series for sufficiently small values of ζ; what makes the Padé approximations useful in our problem is that they provide 'global' analytic approximations to the solution over the entire range $0 \leq \zeta < \infty$, and, moreover, can be used to determine an estimate for the value Φ corresponding to the breather-type solution. The $[2N \times 2N]$ Padé approximant used herein has the form [16],

$$\varphi^{[2N\times 2N])}(\zeta) = \frac{a_0 + a_2\zeta^2 + a_4\zeta^4 + \cdots + a_{2N}\zeta^{2N}}{1 + b_2\zeta^2 + b_4\zeta^4 + \cdots + b_{2N}\zeta^{2N}}\;. \tag{18}$$

Only even powers of ζ are retained due to the anticipated symmetry of the solution, $\varphi(\zeta) = \varphi(-\zeta)$. The coefficients of the rational expression (18) are computed in terms of Φ by imposing the following matching between $\varphi^{[2N\times 2N]}(\zeta)$ and the Taylor series (15a) which is performed correct to $O(\zeta^{4N})$:

$$\frac{a_0 + a_2\zeta^2 + 4\zeta^4 + \cdots + a_{2N}\zeta^{2N}}{1 + b_2\zeta^2 + b_4\zeta^4 + \cdots + b_{2N}\zeta^{2N}} = \sum_{p=0}^{2N} C_{2p}\zeta^{2p} + O(\zeta^{4N+2}) \tag{19}$$

Matching coefficients of respective powers of ζ^{2p} in (19), we obtain the following expressions for the coefficients of the Padé approximation [16]:

$$\begin{Bmatrix} b_2 \\ b_4 \\ \vdots \\ b_{2N} \end{Bmatrix} = \begin{bmatrix} C_{2N} & C_{2N-2} & \ldots & C_6 & C_4 & C_2 \\ C_{2N+2} & C_{2N} & \ldots & C_8 & C_6 & C_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{4N-2} & C_{4N-4} & \ldots & C_{2N+4} & C_{2N+2} C_{2N} & \end{bmatrix}^{-1} \begin{Bmatrix} -C_{2N+2} \\ -C_{2N+4} \\ \vdots \\ -C_{4N} \end{Bmatrix} \tag{20a}$$

and

$$\begin{aligned} a_0 &= C_0, \\ a_2 &= C_2 + b_2 C_0 \\ a_4 &= C_4 + b_2 C_2 + b_4 C_0, \\ &\ldots \\ a_{2N} &= \sum_{j=0}^{N} b_{2(N-j)} C_{2j}. \end{aligned} \tag{20b}$$

Hence, all coefficients of $\varphi^{[2N\times 2N]}(\zeta)$ can be parametrized in terms of Φ, and the Padé approximation (18) becomes a one-parameter family of analytical approximations of the solutions of (14) with initial conditions $(\varphi(0), \varphi'(0)) = (\Phi, 0)$. We now compute the value of Φ for which $\varphi^{[2N\times 2N]}(\zeta)$ decays to zero as ζ tends to infinity. Denoting this value by $\hat{\Phi}^{[2N]}$ and considering the rational structure of (18), the necessary and sufficient condition for the decay of $\varphi^{[2N\times 2N]}(\zeta)$ for large values of ζ is,

$$\lim_{\zeta\to\infty} \varphi^{[2N\times 2N]}(\zeta; \hat{\Phi}^{[2N]}) = 0 \Leftrightarrow a_{2N}(\hat{\Phi}^{[2N]}) = 0 \quad \text{and} \quad b_{2N}(\hat{\Phi}^{[2N]}) \neq 0, \tag{21}$$

where the parametrization of $\varphi^{[2N\times 2N]}$ with respect to Φ is explicitly denoted. For a given order $2N$ of the diagonal Padé approximation relations (21) provide a means to numerically compute $\hat{\Phi}^{[2N]}$. To eliminate mathematical (spurious) solutions, a convergence study is performed by varying the order of the Padé approximation and selecting the converging numerical value of Φ that satisfies the above relations.

We numerically computed the Padé approximants (18) using *Mathematica* up to order $2N = 8$. We then imposed the conditions (21) and obtained the following convergent values of $\hat{\Phi}^{[2N]}$ for varying orders $2N$:

Padé order $2N$	Estimate $\hat{\Phi}^{[2N]}$
2	$\pm\sqrt{3}$
4	± 2.20701
6	± 2.21121
8	± 2.21200

In Figure 1 the decaying Padé approximations $\varphi^{[2N\times 2N]}(\zeta; \hat{\Phi}^{[2N]})$ are graphically depicted, and the convergence of the solution with increasing order N is shown.

To compare the derived analytical approximations with numerical solutions, the Equation (16) was numerically integrated with initial conditions specified at $\zeta = 10^{-8}$ (in order to avoid the singularity at $\zeta = 0$). The initial condition corresponding to the decaying solution was numerically estimated as $\hat{\Phi}^{\text{num}} \approx \pm 2.206208416865$, which compared to $\hat{\Phi}^{[8]}$ indicates a 0.262% error in the analytical estimate. In Figure 2 the approximation $\varphi^{[8\times 8]}(\zeta; \hat{\Phi}^{[8]})$ is compared to the numerical solution; satisfactory agreement is noted.

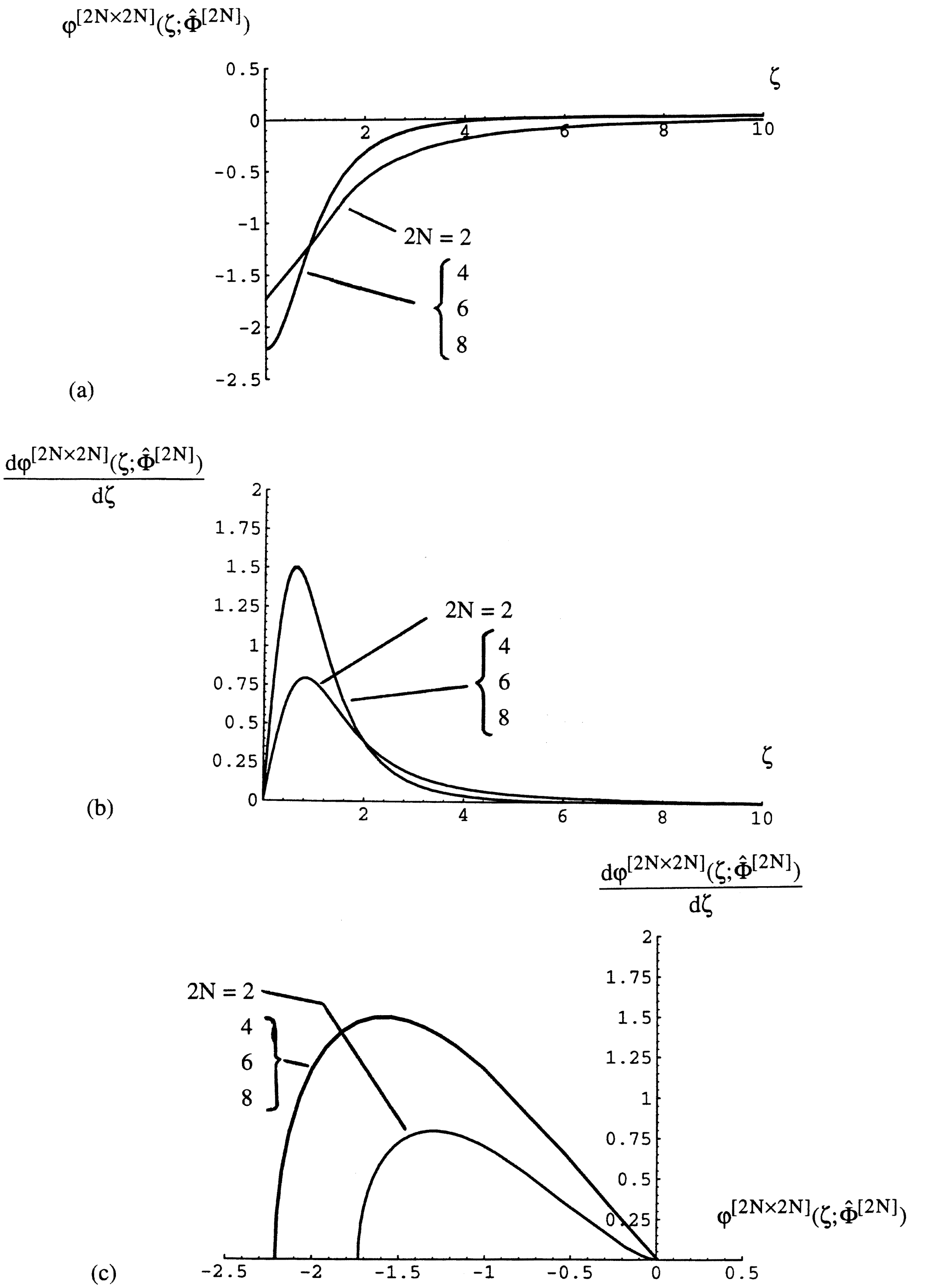

Figure 1. Convergence of the Padé approximations for $2N = 2, 4, 6$ and 8: (a) $\varphi^{[2N\times 2N]}(\zeta;\hat{\Phi}^{[2N]})$ as functions of ζ, (b) $\mathrm{d}\varphi^{[2N\times 2N]}(\zeta;\hat{\Phi}^{[2N]})/\mathrm{d}\zeta$ as functions of ζ, and (c) in the projection of the phase space.

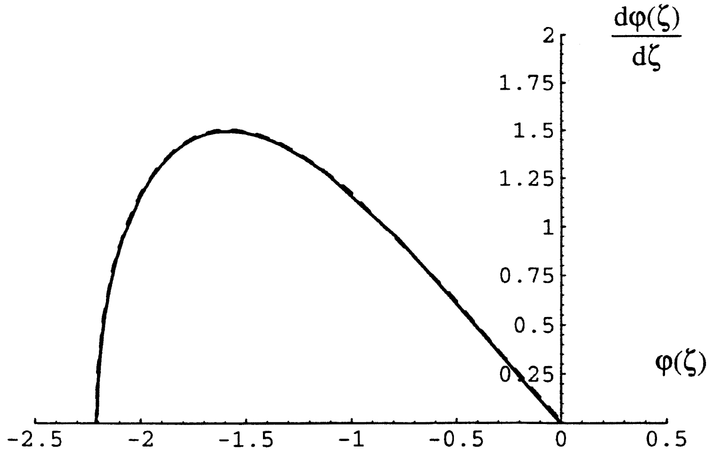

Figure 2. Comparison between $\varphi^{[8\times 8]}(\zeta;\hat{\Phi}^{[8]})$ and the numerical solution of (16) in the projection of the phase space; —— Padé approximation, - - - - - - numerical solution.

From the previous analysis, the solution of problem (13) is approximated as,

$$\varphi(\zeta) = \left\{ \frac{a_0 + a_2\zeta^2 + 4\zeta^4 + a_6\zeta^6}{1 + b_2\zeta^2 + b_4\zeta^4 + b_6\zeta^6 + b_8\zeta^8} \right\}_{\Phi=\hat{\Phi}^{[8]}} + O(\zeta^{18}) \tag{22}$$

with the various Padé coefficients given by (20) and (17b); the corresponding analytical expressions for these coefficients are lengthy nonlinear functions of Φ and are not reproduced here. The scalar κ in Equations (12) and (13) is then computed by the relation,

$$\varphi(0) = \pm\left(\frac{\beta}{\beta+\kappa}\right)^{1/2} = \hat{\Phi}^{[8]} \approx \pm 2.212 \Rightarrow \kappa \approx -0.7956\beta, \tag{23}$$

where β is defined in (12). The $O(1)$ approximation $a_1^{(0)}$ is then evaluated through the last of relations (12).

4. Concluding Remarks

We analyzed axisymmetric standing breathers of the nonlinear partial differential Equation (1). The asymptotic analysis was performed by defining a reference displacement and constructing analytical approximations to the modal function describing the (nonlinear) dependence of the motion on the reference displacement. The leading order approximation of the spatial distribution of the breather is governed by a two-dimensional NLS equation with cubic nonlinearities. Hence, the problem of computing the breather was converted to the problem of analytically approximating the homoclinic orbit of the NLS equation. This was performed by constructing $[2N \times 2N]$ Padé approximants of the sought solution, expressing the Padé coefficients in terms of an initial amplitude, and imposing a necessary and sufficient condition to ensure decay of the Padé approximations as the independent variable tends to infinity. Moreover, a convergence study was performed in order to eliminate additional 'spurious' solutions.

To the best knowledge of the authors the outlined technique for analytically estimating the homoclinic orbit of a nonlinear dynamical system is presented for the first time in the literature, and might provide a tool for computing homoclinic or heteroclinic trajectories of dynamical systems of higher dimensions. Moreover, the methodology for computing axisymmetric breathers presented in this work is sufficiently general to be applicable to other types of nonlinear partial differential equations of dimensions higher than two.

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References

1. Kosevich, A. M. and Kovalev A. S., 'Self-localization of vibrations in a one-dimensional anharmonic chain', *Soviet Physics – JETP* **40**(5), 1975, 891–896.
2. Flytzanis, N., Pnevmatikos, S., and Renoissenet M., 'Kink, breather, and asymmetric envelope or dark solitons in nonlinear chains', *Journal of Physics C: Solid State Physics* **18**, 1985, 4603–4620.
3. Peyrard, M. and Pnevmatikos, S., 'Discreteness effects on non-topological kink soliton dynamics in nonlinear lattices', *Physica D* **19**, 1986, 268–281.
4. Rössler, T. and Page, J. B., 'Intristic localized modes in driven anharmonic lattices with realistic potentials', *Physics Letters A* **204**, 1995, 418–426.
5. Weinstein, A., 'Periodic nonlinear waves on a half-line', *Communications of Mathematical Physics* **99**, 1985, 385–388.
6. Kruskal, M. and Segur, H., 'Nonexistence of small-amplitude breather solutions in ϕ^4-theory', *Physics Review Letters* **58**, 1987, 747.
7. Eleonosky, V., 'Problems of existence of nontopological solitons (breathers) for nonlinear Klein–Gordon equations', in *Proceedings of a NATO Advanced Research Workshop on Asymptotics Beyond all Orders*, La Jolla, CA, 1991, pp. 357–363.
8. King, M. E. and Vakakis, A. F., 'A method for studying waves with spatially localized envelopes in a class of nonlinear partial differential equations', *Wave Motion* **19**, 1994, 391–405.
9. Newton, P. K. and Watanabe, S., 'The geometry of nonlinear Schrödinger standing waves', Center for Complex Systems Research Technical Report UIUC-BI-CCSR-91-08, University of Illinois, Urbana, IL, 1991.
10. Strauss, W., 'Existence of solitary waves in higher dimensions', *Communications of Mathematical Physics* **55**, 1977, 149–162.
11. Berestycki, H. and Lions, P. L., 'Nonlinear scalar field equations II: Existence of infinitely many solutions', *Archive of Rational Mechanics and Analysis* **82**, 1983, 347–376.
12. Ryder, G. H., 'Boundary value problems for a class of nonlinear differential equations', *Pacific Journal of Mathematics* **22**, 1968, 477–503.
13. MacKay, R. S. and Aubry, S., 'Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators', Warwick Preprint 20/1994, University of Warwick, UK, 1994.
14. Akylas, T. R., 'Three-dimensional long water-wave phenomena', *Annual Review of Fluid Mechanics* **26**, 1994, 191–210.
15. Vakakis, A. F., Manevitch, L. I., Mikhlin Yu. V., Pilipchuck, V. N., and Zevin, A. A., *Normal Modes and Localization in Nonlinear Systems*, Wiley Interscience, New York, 1996.
16. Baker Jr., G. A. and Graves-Morris, P., *Padé Approximants I and II*, Encyclopedia of Mathematics and Its Applications, Vols. 13 and 14, Addison-Wesley, Reading, MA, 1981.
17. Sayadi, M. K. and Pouget, J., 'Soliton dynamics in a microstructured lattice model', *Journal of Physics A* **24**, 1991, 2151–2172.
18. Rosenau, P., 'Quasi-continuous spatial motion of a mass-spring chain', *Physica D* **27**, 1987, 224–234.
19. Nayfeh, A. H. and Mook, D. T., *Nonlinear Oscillations*, Wiley Interscience, New York, 1979.
20. Dadfar, M. B., Geer, J., and Andersen, C. M., 'Perturbation analysis of the limit cycle of the free Van der Pol equation', *SIAM Journal on Applied Mathematics* **44**(5), 1984, 881–895.
21. Dadfar, M. B. and Geer, J., 'Power series solution to a simple pendulum with oscillating support', *SIAM Journal on Applied Mathematics* **47**(4), 1987, 737–750.

22. Andrianov, I. V., ‘Application of Padé approximants in perturbation methods’, *Advances in Mechanics* **14**(2), 1991, 3–25.
23. Molotkov, I. A. and Vakulenko, S. A., *Localized Nonlinear Waves*, Leningrad University Press, 1988 [in Russian].
24. Chiao, R. Y., Carmine, E., and Towes, C. H., ‘Self-trapping of optical beams’, *Physics Review Letters* **13**(15), 1964, 479–482.
25. Filkelstein, R., Leverier, R., and Ruderman, M., ‘Nonlinear spinor fields’, *Physics Review Letters*, **83**(4), 1951, 326–332.