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NUMERICAL-ANALYTICAL METHODS OF SOLUTION OF INTEGRAL EQUATIONS ON TWO-DIMENSIONAL DIFFRACTION PROBLEMS

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1. Introduction

In the lecture, three following types of integral equations (I.E.) are considered

$$\int_{-1}^{1} \rho(\xi)[\ln|\xi-\eta| + N_1(\xi,\eta;\varepsilon)]d\xi = f_1(\eta), \quad \eta \in [-1,\ 1] \tag{1}$$

$$\int_{-\alpha}^{\alpha} \rho(\varphi)\left[\ln 2\left|\sin\left(\frac{\varphi-\varphi_0}{2}\right)\right| + N_2(\varphi,\varphi_0;\varepsilon)\right] d\varphi = f_2(\varphi_0), \quad \varphi_0 \in [-\alpha,\ \alpha] \tag{2}$$

$$\int_{0}^{\infty} \rho(x) H_0^{(1)}(\varepsilon|x-x_0|)dx = f_3(x_0), \quad x_0 \in (0,\ \infty) \tag{3}$$

Here $\rho(\xi)$ are the functions to be found; $\{N_j\}_{j=1}^{2}$ and $\{f_j\}_{j=1}^{3}$ are known functions; $H_0^{(1)}(x)$ is the Hankel function of zero order; ε is a real parameter. In addition functions $(N_j, N_j') \in C$ and $(f_j, f_j') \in C$ belong to the class of continuous functions.

Equations (1), (2) are I.E. of the 1-st kind with logarithmic difference kernel. Equation (3) is an I.E. of Wiener–Hopf type. In equation (2) both function $\rho(\varphi)$ and kernel are periodic functions.

There is a lot of interior and exterior boundary-value diffraction problems which may be reduced to equations (1)–(3). E.g., the problems of plane waves diffraction by an infinitely thin strip and non-closed circular cylindrical screen may be reduced to solution of I.E. of the following type

$$\int_{-1}^{1} \rho(\xi) H_0^{(1)}(\varepsilon|\xi-\eta|)d\xi = f_1(\eta), \tag{4}$$

$$\int_{-\alpha}^{\alpha} \rho(\varphi) H_0^{(1)}\left(\varepsilon\left|\sin\left(\frac{\varphi-\varphi_0}{2}\right)\right|\right) d\varphi = f_2(\varphi_0) \tag{5}$$

Here the unknown functions $\rho(\xi)$ and $\rho(\varphi)$ are proportional to current density on the screen, $\varepsilon = ka = 2\pi a/\lambda$ is the frequency parameter, a and α characterize the geometry of the screens. As the function $H_0^{(1)}(x)$ may be represented by

$$H_0^{(1)}(x) = \frac{2i}{\pi}\ln x + N(x), \tag{6}$$

where $N(x)$ is a continuous function, it is clear that equations (4),(5) may be reduced to equations (1),(2).

We shall construct the solutions of I.E. (1)–(3) by means of the method of orthogonal polynomials (O.P.) which is rather straightforward and general one. This method is a particular case of more general scheme of Bubnov's method applied to I.E. but differs by two factors. It needs firstly to investigate preliminary the structure of solution near the edge point of the domain of integration and secondly to construct spectral expressions for singular parts of the kernels with orthogonal polynomials as eigenfunctions.

The method of O.P. is widely used for solution of problems of the theory of elastisity and continuous media mechanics [3–5]. Nevertheless, it has not yet found applications in electromagnetic theory.

2. I.E. with Logarithmic Difference Kernel

At first let us consider I.E. (1). Assume the solution structure to be such that $\rho(\xi)$ satisfies the condition

$$\rho(\xi) \underset{\xi\to\pm1}{\sim} (1-\xi^2)^{-1/2} \tag{7}$$

That means $\rho(\xi)$ has square–root singularity at the ends of interval $[-1,\ 1]$. It is well known that condition (7) follows from Meixner condition in diffraction problems [1, 6].

The following theorem takes place: *If* $f_1'(\eta) \in L_2^{1/2}[-1,1;(1-\eta^2)^{-1/2}]$ *then the unique solution exists of I.E. (1) such that* $\rho(\xi) = (1-\xi^2)^{-1/2}\varphi(\xi)$ *where* $\varphi(\xi) \in L_2^{1/2}[-1,1;(1-\xi^2)^{-1/2}]$.

Here $L_2^{1/2}[-1,1;(1-\eta^2)^{-1/2}]$ *is Hilbert space with scalar product defined as having weight function* $(1-\eta^2)^{-1/2}$.

Let us expand the unknown function $\rho(\xi)$ as follows

$$\rho(\xi) = (1-\xi^2)^{-1/2}\varphi(\xi) = (1-\xi^2)^{-1/2}\sum_{n=0}^{\infty}\rho_n T_n(\xi), \tag{8}$$

where $\{T_n(\xi)\}_{n=0}^{\infty}$ are Chebyshev polynomials of the 1-st kind, $\{\rho_n\}_{n=0}^{\infty}$ are the coefficients to be found.

The expansion (8) for $\rho(\xi)$ is caused by the fact that Chebyshev polynomials $T_n(\xi)$ are eigenfunctions of the following integral operator (I.O.) [3, 5]

$$-\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(\xi)}{\sqrt{1-\xi^2}}\ln|\xi-\eta|d\xi = w_nT_n(\eta), \quad |\eta|\leqslant 1 \tag{9}$$

Here $\{w_n\}_{n=0}^{\infty}$ are eigenvalues of I.O. and are given by

$$w_n = \begin{cases} \ln 2, & n=0 \\ \dfrac{1}{n}, & n=1,\ 2,\ \dots \end{cases} \tag{10}$$

For further derivations we need orthogonality condition for Chebyshev polynomials given by

$$\int_{-1}^{1} T_n(\eta)T_k(\eta)\frac{d\eta}{\sqrt{1-\eta^2}} = \frac{1}{\beta_n}\pi\delta_{kn}; \quad \beta_n = \begin{cases} 1, & n=0 \\ 2, & n\neq 0 \end{cases} \tag{11}$$

Using (9) and (11) one can show the kernel of (9) to have a convergent in certain sense bilinear expansion

$$\frac{1}{\pi}\ln\frac{1}{|\xi-\eta|} = \frac{1}{\pi}\sum_{n=0}^{\infty} w_n T_n(\xi)T_n(\eta) \tag{12}$$

Substituting (8) into I.E. (1), let us take into account spectral expression (9). Then assuming the continuous function to be represented by the series expression

$$f_1(\eta) = \sum_{n=0}^{\infty} f_n T_n(\eta) \tag{13}$$

and using orthogonality condition (11) for seeking coefficients $\{\rho_n\}_{n=0}^{\infty}$, we obtain infinite system of linear algebraic equations (S.L.A.E.)

$$\rho_k - \sum_{n_0}^{\infty} a_{kn}\rho_n = \gamma_k, \quad k = 0,\ 1,\ \ldots \tag{14}$$

Here

$$a_{kn} = \frac{\beta_k}{2w_k}\int_{-1}^{1}\int_{-1}^{1}\frac{T_k(\eta)T_n(\xi)}{\sqrt{(1-\eta^2)(1-\xi^2)}}N_1(\xi,\eta;\varepsilon)d\eta d\xi$$

$$\gamma_k = \pi f_k/(2w_k) \tag{15}$$

It may be shown that $\{\gamma_k\}_{k=0}^{\infty} \in l_2$ and besides

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}|a_{kn}|^2 < \infty \tag{16}$$

It means that matrix $\{a_{kn}\}_{k,n=0}^{\infty}$ produces an absolutely continuous operator A, from l_2 to l_2. Consequently, this operator may be approximated by a finite-dimensional one, in other words S.L.A.E. (14) may be treated by means of truncation.

Note that double integral in matrix elements $\{a_{kn}\}_{k,n=0}^{\infty}$ may be transformed into a single-dimensional one provided that the function $N_1(\xi,\eta;\varepsilon)$ has a representation

$$N_1(\xi,\eta;\varepsilon) = \int_0^{\infty}\delta(\alpha)\cos[\varepsilon(\eta-\xi)]\frac{d\alpha}{\alpha}; \quad \delta(\alpha) \underset{\alpha\to\infty}{\sim} O(\alpha^{-s}),\ s>1 \tag{17}$$

In order to show this let us substitute (17) into (15) and take into account that [7]

$$\int_0^1 \left\{ \begin{matrix} T_{2k}(\eta)\cos(\varepsilon\alpha\eta) \\ T_{2k+1}(\eta)\sin(\varepsilon\alpha\eta) \end{matrix} \right\} \frac{d\eta}{\sqrt{1-\eta^2}} = (-1)^k \frac{\pi}{2} \left\{ \begin{matrix} J_{2k}(\varepsilon\alpha) \\ J_{2k+1}(\varepsilon\alpha) \end{matrix} \right\} \tag{18}$$

(Here $J_k(x)$ are Bessel functions).

Then for a_{kn} we obtain

$$a_{kn} \sim \frac{[1+(-1)^{k+n}]}{w_k} \beta_k \int_0^\infty \frac{\delta(\alpha)}{\alpha} J_k(\varepsilon\alpha) J_n(\varepsilon\alpha) d\alpha \tag{19}$$

In practice [5], the given method of solution of I.E. (1) may be shown to be effective only for small values of ε ($\varepsilon \sim 0 \div 10$). As a rule, this parameter is related with some wave dimensions of scatterers. It may be illustrated by I.E. (4), (5) describing plane waves diffraction problems by cylindrical screens.

3. Method of O.P., Effective for Arbitrary Values of ε

Now we shall give the alternative method of solution of I.E. (1). This algorithm is based also on O.P. method but it is effective for all values of ε including the large values of ε.

Let us make the transformation of variables in I.E. as following

$$\eta = \frac{\operatorname{th}\left(\frac{\varepsilon}{2}\xi\right)}{\operatorname{th}\left(\frac{\varepsilon}{2}\right)} = \alpha; \quad \xi = \frac{\operatorname{th}\left(\frac{\varepsilon}{2}\xi_0\right)}{\operatorname{th}\left(\frac{\varepsilon}{2}\right)} = \beta \tag{20}$$

Then the kernel of I.E. takes the form

$$\ln|\xi-\eta| = \ln\left|\frac{\operatorname{sh}\frac{\varepsilon}{2}(\xi-\xi_0)}{\operatorname{ch}\frac{\varepsilon}{2}\xi \cdot \operatorname{ch}\frac{\varepsilon}{2}\xi_0}\right| - \ln\left(\operatorname{th}\frac{\varepsilon}{2}\right) \tag{21}$$

Now the unknown function $\rho(\alpha)$ is represented by

$$\rho(\alpha) = \frac{\operatorname{ch}\frac{\varepsilon}{2}\xi}{\sqrt{2(\operatorname{ch}\varepsilon - \operatorname{ch}\varepsilon\xi)}} \sum_{n=0}^{\infty} x_n T_n(\alpha) \tag{22}$$

Expansion into series (22) follows from the fact that Chebyshev polynomials $T_n(\alpha)$ are eigenfunctions of I.O. with the kernel (21), and the following spectral expression takes place [3]

$$-\frac{1}{\pi}\int_{-1}^{1} \ln\left|\frac{\operatorname{sh}\frac{\varepsilon}{2}(\xi-\xi_0)}{\operatorname{ch}\frac{\varepsilon}{2}\xi \cdot \operatorname{ch}\frac{\varepsilon}{2}\xi_0}\right| \frac{T_n(\alpha)}{\sqrt{2(\operatorname{ch}\varepsilon - \operatorname{ch}\varepsilon\xi)}} \frac{d\xi}{\operatorname{ch}\frac{\varepsilon}{2}\xi} = \lambda_n T_n(\alpha) \tag{23}$$

Here $\{\lambda_n\}_{n=0}^{\infty}$ are eigenvalues of I.O. equal to

$$\lambda_n = \begin{cases} \dfrac{1}{\varepsilon \operatorname{ch}\frac{\varepsilon}{2}} \ln\left(2\operatorname{cth}\frac{\varepsilon}{2}\right), & n=0 \\ \dfrac{1}{n\varepsilon \operatorname{ch}\frac{\varepsilon}{2}}, & n=1,\ 2,\ \ldots \end{cases} \tag{24}$$

Besides it should be noted that Chebyshev polynomials satisfy orthogonality expression

$$\int_{-1}^{1} \frac{T_n(\alpha)T_k(\alpha)}{\sqrt{2(\operatorname{ch}\varepsilon - \operatorname{ch}\varepsilon\xi)}} \frac{d\xi}{\operatorname{ch}\frac{\varepsilon}{2}\xi} = \frac{1}{\beta_n}\frac{\pi}{\varepsilon \operatorname{ch}\frac{\varepsilon}{2}}\delta_{kn} \tag{25}$$

Now let us substitute (22) into I.E. (1) and take into account spectral expansion (23). Using orthogonality condition (25) for seeking the coefficients $\{x_n\}_{n=0}^{\infty}$, we obtain infinite S.L.A.E.

$$x_k + \sum_{n=0}^{\infty} c_{kn}x_n = q_k, \quad k = 0,\ 1,\ \ldots \tag{26}$$

Here

$$c_{kn} = \frac{\varepsilon\beta_k \operatorname{ch}\frac{\varepsilon}{2}}{8\pi\lambda_k}\int_{-1}^{1}\int_{-1}^{1} \frac{N_1(\alpha,\beta;\varepsilon)T_k(\alpha)T_n(\beta_n)}{\sqrt{(\operatorname{ch}\varepsilon - \operatorname{ch}\varepsilon\xi)(\operatorname{ch}\varepsilon - \operatorname{ch}\varepsilon\xi_0)}}\frac{d\xi d\xi_0}{\operatorname{ch}\frac{\varepsilon}{2}\xi \cdot \operatorname{ch}\frac{\varepsilon}{2}\xi_0} +$$
$$+\delta_{k0}\delta_{n0}\frac{\ln \operatorname{th}\left(\frac{\varepsilon}{2}\right)}{\ln\left(2\operatorname{cth}\frac{\varepsilon}{2}\right)}, \tag{27}$$

$$q_k = -\frac{\beta_k}{2\lambda_k}\operatorname{sh}\frac{\varepsilon}{2}\int_{-1}^{1}\frac{f_1(\beta)T_k(\beta)}{\sqrt{2(\operatorname{ch}\varepsilon - \operatorname{ch}\varepsilon\xi_0)}}\frac{d\xi_0}{\operatorname{ch}\frac{\varepsilon}{2}\xi_0}. \tag{28}$$

It may be shown that number sequence $\{q_k\}_{k=0}^{\infty} \in l_2$, and operator C produced by the matrix $\{c_{kn}\}_{k,n=0}^{\infty}$ operates absolutely continuously from l_2 to l_2. That is why it may be approximated by a finite-dimensional operator.

4. I.E. with a Periodic-Difference Logarithmic Kernel.

Let us consider now I.E. (2). Assume again that unknown function $\rho(\varphi)$ satisfies the condition

$$\rho(\varphi) \underset{\varphi\to\pm\alpha}{\sim} (\varphi^2 - \alpha^2)^{-1/2} \tag{29}$$

For the sake of convenience we represent functions $\rho(\varphi)$ and $f_2(\varphi)$ as the sum of even and odd terms

$$\rho(\varphi) = \frac{1}{2}[\rho_+(\varphi) + \rho_-(\varphi)]; \quad f_2(\varphi_0) = \frac{1}{2}[f_+(\varphi_0) + f_-(\varphi_0)]$$
$$\rho_\pm(\varphi) = \rho(\varphi) \pm \rho(-\varphi); \quad f_\pm(\varphi_0) = f_2(\varphi_0) \pm f_2(-\varphi_0) \tag{30}$$

Then I.E. for the functions $\rho_\pm(\varphi)$ are given

$$\int_{-\alpha}^{\alpha} \rho_\pm(\varphi)\left[-\ln\frac{1}{2\left|\sin\frac{\varphi-\varphi_0}{2}\right|} + N_2(\varphi,\varphi_0;\varepsilon)\right]d\varphi = f_\pm(\varphi_0); \quad \varphi_0 \in [-\alpha,\ \alpha] \tag{31}$$

For the functions $\rho_{\pm}(\varphi)$ we can write

$$\rho_+(\varphi) = \frac{\cos\frac{\varphi}{2}}{\sqrt{2(\cos\varphi - \cos\alpha)}} \sum_{n=0}^{\infty} x_{2n} T_{2n}\left(\frac{\sin\frac{\varphi}{2}}{\sin\frac{\alpha}{2}}\right), \quad \varphi \in [-\alpha,\ \alpha] \tag{32}$$

$$\rho_-(\varphi) = \frac{1}{\cos\frac{\varphi}{2}\sqrt{2(\cos\varphi - \cos\alpha)}} \sum_{n=0}^{\infty} x_{2n-1} T_{2n-1}\left(\frac{\operatorname{tg}\frac{\varphi}{2}}{\operatorname{tg}\frac{\alpha}{2}}\right), \quad \varphi \in [-\alpha,\ \alpha] \tag{33}$$

As before, the form of expansion (32), (33) is caused by the fact that Chebyshev polynomials $T_{2n}\left(\sin\frac{\varphi}{2}/\sin\frac{\alpha}{2}\right)$ and $T_{2n-1}\left(\operatorname{tg}\frac{\varphi}{2}/\operatorname{tg}\frac{\alpha}{2}\right)$ are eigenfunctions of I.O. corresponding to the singular part of the I.E. kernel (31). As has been shown in [4], there exist the following spectral expressions

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \ln\frac{1}{2\left|\sin\frac{\varphi-\varphi_0}{2}\right|} \frac{T_{2n}\left(\frac{\sin\varphi/2}{\sin\alpha/2}\right)}{\sqrt{2(\cos\varphi - \cos\alpha)}} \cos\frac{\varphi}{2}\, d\varphi = \sigma_{2n} T_{2n}\left(\frac{\sin\frac{\varphi_0}{2}}{\sin\frac{\alpha}{2}}\right) \tag{34}$$

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \ln\frac{1}{2\left|\sin\frac{\varphi-\varphi_0}{2}\right|} \frac{T_{2n-1}\left(\frac{\operatorname{tg}\varphi/2}{\operatorname{tg}\alpha/2}\right)}{\sqrt{2(\cos\varphi - \cos\alpha)}} \frac{d\varphi}{\cos\frac{\varphi}{2}} = \sigma_{2n-1} T_{2n-1}\left(\frac{\operatorname{tg}\frac{\varphi_0}{2}}{\operatorname{tg}\frac{\alpha}{2}}\right) \tag{35}$$

Here $\{\sigma_n\}_{n=0}^{\infty}$ are eigenvalues of I.O., equal to

$$\sigma_{2k} = \begin{cases} -\ln\sin\frac{\alpha}{2}, & n = 0 \\ \frac{1}{2n}, & n = 1,\ 2,\ \ldots \end{cases}; \quad \sigma_{2n-1} = \frac{1}{(2n-1)\cos\frac{\alpha}{2}}, \quad n = 1,\ 2,\ \ldots \tag{36}$$

Besides it should be noted that for Chebyshev polynomials $T_n\left(\sin\frac{\varphi}{2}/\sin\frac{\alpha}{2}\right)$ and $T_n\left(\operatorname{tg}\frac{\varphi}{2}/\operatorname{tg}\frac{\alpha}{2}\right)$ the following orthogonality conditions are valid

$$\begin{aligned} &\int_{-\alpha}^{\alpha} T_n\left(\frac{\sin\frac{\varphi}{2}}{\sin\frac{\alpha}{2}}\right) T_k\left(\frac{\sin\frac{\varphi}{2}}{\sin\frac{\alpha}{2}}\right) \frac{\cos\frac{\varphi}{2}}{\sqrt{(\cos\varphi - \cos\alpha)\cdot 2}}\, d\varphi = \frac{1}{\beta_n}\pi\delta_{nk}, \\ &\int_{-\alpha}^{\alpha} T_n\left(\frac{\operatorname{tg}\frac{\varphi}{2}}{\operatorname{tg}\frac{\alpha}{2}}\right) T_k\left(\frac{\operatorname{tg}\frac{\varphi}{2}}{\operatorname{tg}\frac{\alpha}{2}}\right) \frac{d\varphi}{\cos\frac{\varphi}{2}\sqrt{(\cos\varphi - \cos\alpha)\cdot 2}} = \frac{\pi}{\beta_n\cos\frac{\alpha}{2}}\delta_{nk} \end{aligned} \tag{37}$$

The kernels of I.E. (34), (35) may be shown to have the following bilinear expansions

$$\ln\frac{1}{2\left|\sin\frac{\varphi-\varphi_0}{2}\right|} = \begin{cases} \sum_{n=0}^{\infty} \sigma_{2n}\beta_n T_{2n}\left(\frac{\sin\frac{\varphi}{2}}{\sin\frac{\alpha}{2}}\right) T_{2n}\left(\frac{\sin\frac{\varphi_0}{2}}{\sin\frac{\alpha}{2}}\right) \\ 2\sum_{n=0}^{\infty} \sigma_{2n-1} T_{2n-1}\left(\frac{\operatorname{tg}\frac{\varphi}{2}}{\operatorname{tg}\frac{\alpha}{2}}\right) T_{2n-1}\left(\frac{\operatorname{tg}\frac{\varphi_0}{2}}{\operatorname{tg}\frac{\alpha}{2}}\right) \end{cases} \tag{38}$$

Now let us substitute expressions (32), (33) into I.E. (31). Taking into account spectral expressions (34), (35) and orthogonality (37), we obtain infinite S.L.A.E. for seeking unknown coefficients

$$x_{2k} + \sum_{n=0}^{\infty} b_{2k\ 2n} x_{2n} = f_{2k}^{+}, \qquad k = 0,\ 1,\ \ldots \tag{39}$$

$$x_{2k-1} + \sum_{n=1}^{\infty} b_{2k-1\ 2n-1} x_{2n-1} = f_{2k-1}^{-}, \quad k = 1,\ 2,\ \ldots \tag{40}$$

Here we have denoted

$$b_{2k\ 2n} =$$

$$= \frac{\beta_k}{8\sigma_{2k}} \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{\cos\frac{\varphi}{2}\cos\frac{\varphi_0}{2} T_{2k}\left(\frac{\sin\varphi/2}{\sin\alpha/2}\right) T_{2n}\left(\frac{\sin\varphi_0/2}{\sin\alpha/2}\right)}{\sqrt{(\cos\varphi - \cos\alpha)(\cos\varphi_0 - \cos\alpha)}} N_2(\varphi, \varphi_0; \varepsilon)\, d\varphi\, d\varphi_0$$

$$b_{2k-1\ 2n-1} =$$

$$= \frac{2k-1}{4}\cos^2\frac{\alpha}{2} \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{T_{2k-1}\left(\frac{\operatorname{tg}\varphi/2}{\operatorname{tg}\alpha/2}\right) T_{2n-1}\left(\frac{\operatorname{tg}\varphi_0/2}{\operatorname{tg}\alpha/2}\right)}{\sqrt{(\cos\varphi - \cos\alpha)(\cos\varphi_0 - \cos\alpha)}} \frac{N_2(\varphi, \varphi_0; \varepsilon)}{\cos\frac{\varphi}{2}\cos\frac{\varphi_0}{2}} d\varphi\, d\varphi_0 \tag{41}$$

$$f_{2k}^{+} = \frac{\beta_k}{4\sigma_{2k}} \int_{-\alpha}^{\alpha} \frac{f_{+}(\varphi_0)\cos\frac{\varphi_0}{2}}{\sqrt{2(\cos\varphi_0 - \cos\alpha)}} T_{2k}\left(\frac{\sin\varphi_0/2}{\sin\alpha/2}\right) d\varphi_0$$

$$f_{2k}^{+} = \frac{2k-1}{2}\cos^2\frac{\alpha}{2} \int_{-\alpha}^{\alpha} \frac{f_{-}(\varphi_0) T_{2k-1}\left(\frac{\operatorname{tg}\varphi_0/2}{\operatorname{tg}\alpha/2}\right)}{\cos\frac{\varphi_0}{2}\sqrt{2(\cos\varphi_0 - \cos\alpha)}} d\varphi_0 \tag{42}$$

It may be shown that

$$\sum_{k=0}^{\infty} |f_{2k}^{+}|^2 < \infty; \quad \sum_{k=1}^{\infty} |f_{2k-1}^{-}|^2 < \infty$$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} |b_{2k\ 2n}|^2 < \infty; \quad \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} |b_{2k-1\ 2n-1}|^2 < \infty \tag{43}$$

In other words, operator B produced by the matrix $\{b_{kn}\}_{k,n=0}^{\infty}$ is an absolutely continuous one and may be approximated by finite–dimensional operator. Then the solution of infinite S.L.A.E. (39), (40) may be obtained with any given accuracy by the truncation method.

As for the validity of truncation method applied to the solution of infinite S.L.A.E. (14), (26), (39), (40), we note the folowing. In paper by G.Ya. Popov [3], the infinite S.L.A.E. resulting from I.E. (1), (2) by the method of O.P. may e always solved by the truncation method provided that the eigenvalues of I.E. connected to the singular part of the kernel have the asymptotics

$$\sigma_n \underset{n\to\infty}{\sim} O(n^{-\gamma}), \quad 0 \leqslant \gamma < 2 \tag{44}$$

It is easy to note that for the eigenvalues of I.O. under investigation there is an asymptotic w_n, λ_n, $\sigma_n \underset{n\to\infty}{\sim} O(n^{-1})$. Thus the condition (44) is satisfied.

To compute the integrals in matrix elements (see (15), (27), (41)) with Chebyshev polynomials one has to make use of quadrature formula of Faylon [3]. The latter gives the best approximation for these integrals.

In general, one may use quadrature formula proposed by G.Ya. Popov for integrals given by [3]

$$J_m = \int_0^1 t^\alpha (1-t)^\beta P_m^{(\alpha,\beta)}(1-2t) f(t)dt, \tag{45}$$

where $\{P_m^{(\alpha,\beta)}(x)\}_{m=0}^{\infty}$ are Jacoby polynomials.

This quadrature formula does not need the values of zeros of Jacoby polynomials and takes into account the oscillation of integrand. All the information on this formula may be found in [3].

5. I.E. with Difference Kernel on Semi-Infinite Interval

Finally, let us consider the particular case of I.E. (3) appearing in the problem of E-polarized plane wave $E_z^0 = \exp[ik(\alpha_0 x + \sqrt{1-\alpha_0^2} y)]$ ($k = 2\pi/\lambda$, $\alpha_0 = \cos\theta$, θ angle of incidence) diffraction by an infinitely thin perfectly conducting halfplane. It may be written as

$$\int_0^\infty \rho(x_0) k_0(k'|x - x_0|) dx_0 = \pi e^{-k'\alpha_0 x} \tag{46}$$

Here $k' = -ik$, $k_0(x) = \frac{\pi i}{2} H_0^{(1)}(ix)$ is McDonald function, $\rho(x_0)$ is surface current density.

Let us show that basing on O.P. method the solution of I.E. (46) may be obtained analytically.

Function $\rho(x_0)$ satisfies the condition

$$\rho(x_0) \underset{x_0\to 0}{\sim} O(x_0^{-1/2}) \tag{47}$$

Introducing the dimensionless parameter $\eta_0 = x_0 k'$, the function $\rho(\eta_0/k')$ may be expanded as

$$\rho(\eta_0/k') = \frac{e^{-\eta_0}}{\sqrt{\eta_0}} \sum_{n=0}^{\infty} \rho_n L_n^{-1/2}(2\eta_0) \tag{48}$$

Here $\{L_n^{-1/2}(2\eta_0)\}_{n=0}^{\infty}$ are Laguerr polynomials, $\{\rho_n\}_{n=0}^{\infty}$ are the unknown coefficients.

This expression (48) follows from the fact that Laguerr polynomials $L_n^{-1/2}(2\eta)$ are eigenfunctions of I.O. [3, 5]

$$\int_0^\infty \frac{k_0(|x-y|)}{\sqrt{y}e^y} L_n^{-1/2}(2y)dy = \frac{\pi}{\sqrt{2}}\gamma_n L_n^{-1/2}(2x)e^{-x}, \tag{49}$$

while the eigenvalues of I.O. are given by

$$\gamma_n = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \underset{n\to\infty}{\sim} O(n^{-1/2}) \tag{50}$$

Orthogonality conditions for Laguerr polynomials are given by

$$\int_0^\infty \frac{e^{-\eta}}{\sqrt{\eta}} L_n^{-1/2}(\eta) L_k^{-1/2}(\eta) d\eta = \gamma_n \delta_{nk} \tag{51}$$

From spectral expression (49) it follows that for the kernel of I.E. (46) the bilinear expansion is valid

$$\begin{aligned} k_0(|x-y|) &= \frac{\pi}{\sqrt{2}} e^{-(x+y)} \sum_{n=0}^{\infty} \gamma_n L_n^{-1/2}(2x) L_n^{-1/2}(2y), \\ iH_0^{(1)}(k|x-y|) &= \sqrt{2} e^{ik(x+y)} \sum_{n=0}^{\infty} \gamma_n L_n^{-1/2}(-2ikx) L_n^{-1/2}(-2iky). \end{aligned} \tag{52}$$

Let us substitute (48) into I.E. (46). Then taking into account spectral expression (49) and orthogonality (51) under the condition $1+\alpha_0 > 0$ we obtain unknown coefficients $\{\rho_n\}_{n=0}^{\infty}$ as follows

$$\rho_n = 2k' \frac{1}{\gamma_n} \frac{(\alpha_0 - 1)^n}{(\alpha_0+1)^{n+1/2}}, \quad \alpha_0 = \cos\theta \tag{53}$$

Thus, we have obtained the function $\rho(\eta_0/k')$ analytically

$$\rho\left(\frac{\eta_0}{k'}\right) = \frac{2k' e^{-\eta_0}}{\sqrt{\eta(1+\alpha_0)}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{\alpha_0-1}{\alpha_0+1}\right)^n L_n^{-1/2}(2\eta_0) \tag{54}$$

Unfortunately, this formula is not valid for arbitrary angles of incidence, as the series converges only if $0 \leqslant \theta < \frac{\pi}{2}$. Nevertheless, this difficulty may be overcame by means of summation using the known formula [7]

$$\begin{aligned} \rho\left(\frac{\eta}{k'}\right) &= \frac{e^{-\eta}}{\sqrt{\pi\eta}} k'(1+\alpha_0)\, {}_1F_1(1; \frac{1}{2}; (1-\alpha_0)\eta) = \\ &= \frac{e^{-\eta}}{\sqrt{\pi\eta}} k'(1+\alpha_0) \left\{ 1 - \sqrt{\pi\eta(1-\alpha_0)} e^{(1-\alpha_0)\eta} \left[1 - \mathrm{erf}\sqrt{\eta(1-\alpha_0)} \right] \right\} \end{aligned} \tag{55}$$

Here ${}_1F_1(a;b;t)$ is the degenerated hypergeometric function, $\mathrm{erf}(x)$ is the probability integral.

Note that representation (55) for surface current density coincides with the results by other authors [8] obtained by means of other methods.

Using representation (55) for $\rho(x)$ one may come to the following asymptotics

$$\text{a)}\quad x \to 0;\quad \rho(x) \approx \sqrt{\frac{2}{\pi kx}}\, e^{i\left(kx-\frac{\pi}{4}\right)} k\cos\frac{\theta}{2}\left[1 - 4ikx\sin^2\frac{\theta}{2}\right] + O\left[(kx)^{3/2}\right]$$

$$\text{b)}\quad x \to \infty;\quad \rho(x) \approx -ik\sin\theta e^{ikx\cos\theta}\left[1 + O\left(\frac{1}{kx}\right)\right]$$

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