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FRACTIONAL *CURL* OPERATOR IN REFLECTION PROBLEMS

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Abstract – **The purpose of this paper is to investigate some applications of the fractionalized *curl* operator in the scattering problems. Using the technique of fractionalizing a linear operator we obtain the presentation for the fractional *curl* operator for the functions of two variables expressed via exponents. We use this presentation in a plane wave reflection at oblique incidence on an impedance infinite surface. Applying fractional *curl* operator to a fixed solution we obtain the fields that describe, in certain sense, an intermediate of "fractional" solution between the original solution and its dual solution. These "fractional" fields represent the solution of reflection problem from an anisotropic surface. The relation between the impedance of such an anisotropic surface and the original impedance are presented in this paper.**

I. INTRODUCTION

In recent years, it has been studied the possible applications of tools of fractional calculus in electromagnetics problems. The idea of fractional derivatives and integrals can be extended to fractionalization of some other operators commonly used in electromagnetics. In [1] it was first proposed a fractionalization of the *curl* operator. New operator $curl^{\alpha}$ was introduced where parameter $0<\alpha<1$ is real. If $\alpha=0$, one gets the identity operator. If $\alpha=1$, one gets the conventional curl operator. In [1] the presentation of the fractional *curl* operator was obtained in explicit form for a function of one variable. But in certain reflection problems we should extend it for the functions of two variables. Following the technique proposed in [1], we shall obtain the fractional *curl* operator when the function is represented via exponents $e^{iax+iby}$. Such functions characterize obliquely incident plane waves. This presentation will be used in the problem of a plane wave incidence on the boundary. We are going to study the fractional fields obtained by applying fractional *curl* operator to the solution of the problem with fixed impedance. We shall also generalize impedance boundary conditions with the aid of the fractional *curl* operator.

II. DEFINITION OF FRACTIONAL *CURL* OPERATOR

The operator "*curl*" is one of the commonly used operators in electrodynamics. Using the spatial Fourier transform from (*x,y,z*)-space into *k*-domain (k_x,k_y,k_z), the *curl* operator of a three-dimensional vector field $\vec{F}=F_x\vec{x}+F_y\vec{y}+F_z\vec{z}$ can be expressed as a cross product of vector $\vec{k}$ with the vector $F_k(\vec{F})$ in *k*-domain, that is $F_k(curl\vec{F})(k_x,k_y,k_z)\equiv i\vec{k}\times F_k(\vec{F})$.

Consider a general linear operator L that acts within the space C^n of *n*-dimensional vectors. The new operator L^{α} is considered as the fractionalized operator (from L) if [1]

1) if $\alpha=1: L^{\alpha}|_{\alpha=1}=L$; 2) if $\alpha=0: L^{\alpha}|_{\alpha=0}=I$ - the identity operator; 3) $L^{\alpha}L^{\beta}=L^{\beta}L^{\alpha}=L^{\alpha+\beta}$.

Operator L^{α} can be defined as operator that has the same eigenvectors $\{\vec{A}_m\}$ as operator L has, but with the eigenvalues of $\{(a_m)^{\alpha}\}$, where $\{a_m\}$ are the eigenvalues of L. An arbitrary vector $\vec{H}$ can be represented as a linear combination of eigenvectors $\vec{A}_m$ with some coefficients, i.e., as $\vec{H}=\sum_{m=1}^{n}g_m\vec{A}_m$. We define operator L^{α} via the action on an arbitrary vector $\vec{H}$ as follows:

$$L^{\alpha}\vec{H}=\sum_{m=1}^{n}(a_m)^{\alpha}g_m\vec{A}_m \tag{1}$$

So we can obtain the operator L^{α} from the knowledge of operator L and its eigenvectors and eigenvalues.

For a fixed vector $\vec{k}$ the operator $(i\vec{k}\times)$ is a linear operator in k-domain that acts on an arbitrary vector $\vec{P}$ as $\vec{S}=(ik\times)(\vec{P})\equiv ik\times\vec{P}$. So if we have a fractionalized cross product operator $(i\vec{k}\times)^{\alpha}$ then we can obtain the fractionalized *curl* operator $curl^{\alpha}$ by applying the inverse Fourier transform. That is $curl^{\alpha}=F_k^{-1}((i\vec{k}\times)^{\alpha})$.

When three-dimensional vector $\vec{F}$ is a function of the z-coordinate only: $\vec{F} = \vec{F}(z)$, then we can obtain the presentation for the fractionalization of cross product using described procedure, and after applying the inverse Fourier transform we obtain the fractional *curl* operator of the vector $\vec{F} = \vec{F}(z)$ [1]:

$$curl^{\alpha}\vec{F}(z) = [\cos(\pi\alpha/2)D_z^{\alpha}F_x(z) - \sin(\pi\alpha/2)D_z^{\alpha}F_y(z)]\vec{x} + [\sin(\pi\alpha/2)D_z^{\alpha}F_x(z) + \cos(\pi\alpha/2)D_z^{\alpha}F_y(z)]\vec{y} + \delta_{0\alpha}D_z^{\alpha}F_z(z)$$

Here the symbol $D_z^{\alpha}f(z) = 1/\Gamma(-\alpha)\cdot\int_{-\infty}^{z}(z-t)^{-\alpha-1}f(t)dt$ is the Riemann-Liouville fractional integral [2].

Now we extend the presentation for the fractional *curl* operator of the vector $\vec{E} = F(x,y)\vec{z}$ for the case when function $F(x,y)$ is expressed via exponents as $F(x,y) = e^{iax+iby}$. It is known that the Fourier transform of exponent can be expressed via the Dirac delta function: $F_k(e^{iax}) = \delta(k_x - a)$. Following the above technique for obtaining fractionalized operator we get the presentation for the fractional *curl* operator of vector $\vec{E} = F(x,y)\vec{z}$:

$$curl^{\alpha}(\vec{z}e^{iax+iby}) = \frac{i^{\alpha}b\sin(\pi\alpha/2)}{(a^2+b^2)^{(1-\alpha)/2}}e^{iax+iby}\vec{x} - \frac{i^{\alpha}a\sin(\pi\alpha/2)}{(a^2+b^2)^{(1-\alpha)/2}}e^{iax+iby}\vec{y} + i^{\alpha}\cos(\pi\alpha/2)\cdot(a^2+b^2)^{\alpha/2}e^{iax+iby}\vec{z} \quad (2)$$

The value α can be, in general, complex and we must consider possible multivalued behavior in the presentation of $curl^{\alpha}$. The choice of appropriate branch is based on physical conditions.

Consider the source-free Maxwell equations

$$(ik_0)^{-1}curl(\eta_0\vec{H}) = -\vec{E}; \quad (ik_0)^{-1}curl\vec{E} = \eta_0\vec{H} \quad (3)$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the intrinsic impedance of the medium.

Using fractional *curl* we can derive a new set of solutions of Maxwell's equations in the following form [1]:

$$\vec{E}^{\alpha} = (ik_0)^{-\alpha}curl^{\alpha}\vec{E}; \quad \eta_0\vec{H}^{\alpha} = (ik_0)^{-\alpha}curl^{\alpha}(\eta_0\vec{H}) \quad (4)$$

If $\alpha = 0$ or $\alpha = 1$, we get the original $\vec{E}^{\alpha}|_{\alpha=0} = \vec{E}, \eta_0\vec{H}^{\alpha}|_{\alpha=0} = \eta_0\vec{H}$ or the dual fields $\vec{E}^{\alpha}|_{\alpha=1} = \eta_0\vec{H}, \eta_0\vec{H}^{\alpha}|_{\alpha=1} = -\vec{E}$. Therefore, fields (4) can be considered as "intermediate" solutions between the original fields and the dual fields. In [1] it was suggested the name "fractional dual fields" for these fields.

III. APPLICATION IN REFLECTION PROBLEMS

Consider a well-known problem of an oblique plane wave incidence on an impedance surface located at the plane $x-z$. The domain $y>0$ is free half-space. Suppose that the incident field is a sum of linearly polarized TE and TM uniform plane waves given by:

$$\vec{E} = \vec{z}e^{ik(x\cos\varphi + y\sin\varphi)}, \quad \vec{H} = -\vec{z}e^{ik(x\cos\varphi+y\sin\varphi)} \quad (5)$$

where φ is the angle between the axis x and the plane of incidence.

The total field is the sum of the incident and reflected fields, $\vec{E}^t = \vec{E}^i + \vec{E}^r$, $\vec{H}^t = \vec{H}^i + \vec{H}^r$, where the reflected field is expressed in terms of reflection coefficients R_{TE}, R_{TM} as

$$\vec{E}^r = \vec{z}R_{TE}e^{ik(x\cos\varphi - y\sin\varphi)}, \quad \vec{H}^r = -\vec{z}R_{TM}e^{ik(x\cos\varphi-y\sin\varphi)} \quad (6)$$

We use impedance boundary conditions (IBC) for an isotropic material in the following form [3]:

$$\vec{n}\times\vec{E} = -\eta\vec{n}\times(\vec{n}\times\eta_0\vec{H}), \; y\to+0 \quad (7)$$

where $\vec{n} = \vec{y}$ is the vector normal to the surface and η is the dimensionless impedance defined as $\eta = \sqrt{\mu/\varepsilon}/\eta_0$ where μ, ε are the permittivity and permeability respectively. For these boundary conditions, the reflection coefficients can be written as [3]

$$R_{TE} = -\frac{1-\eta\sin\varphi}{1+\eta\sin\varphi}, \quad R_{TM} = -\frac{1-\eta^{-1}\sin\varphi}{1+\eta^{-1}\sin\varphi} \quad (8)$$

Consider "fractional" fields given in (4). Using (2) we get the presentations for the fields $\vec{E}^{\alpha}, \vec{H}^{\alpha}$. Then we split these fields into two components - TE and TM parts: $\vec{E}^{\alpha} = \vec{E}_e^{\alpha} + \vec{E}_m^{\alpha}$, $\vec{H}^{\alpha} = \vec{H}_e^{\alpha} + \vec{H}_m^{\alpha}$, where

$$\vec{E}_e^{\alpha} = \vec{z}A^{-}e^{ik_0(x\cos\varphi+y\sin\varphi)} + \vec{z}(R_{TE}\cos(\pi\alpha/2) - R_{TM}\sin(\pi\alpha/2))e^{ik_0(x\cos\varphi - y\sin\varphi)} \quad (9)$$

$$\vec{H}_m^{\alpha} = -\vec{z}A^{+}e^{ik_0(x\cos\varphi+y\sin\varphi)} - \vec{z}(R_{TE}\sin(\pi\alpha/2) + R_{TM}\cos(\pi\alpha/2))e^{ik_0(x\cos\varphi-y\sin\varphi)} \quad (10)$$

$$A^{\pm} = \cos(\pi\alpha/2) \pm \sin(\pi\alpha/2).$$

This set of solutions satisfies the Maxwell equations and represents uniform plane wave. It can be shown that reflection coefficients for these fields are expressed through original reflection coefficients R_E, R_H as follows:

$$R_\alpha^{TE} = \frac{R_{TE}\cos(\pi\alpha/2) - R_{TM}\sin(\pi\alpha/2)}{\cos(\pi\alpha/2) - \sin(\pi\alpha/2)}, \quad R_\alpha^{TM} = \frac{R_{TM}\cos(\pi\alpha/2) + R_{TE}\sin(\pi\alpha/2)}{\cos(\pi\alpha/2) + \sin(\pi\alpha/2)} \tag{11}$$

By using these reflection coefficients, the fields (9),(10) can be decomposed into the incident field ($\vec{E}_e^{\alpha,i}, \vec{H}_m^{\alpha,i}$) and the reflected field ($\vec{E}_e^{\alpha,r}, \vec{H}_m^{\alpha,r}$) as:

$$\vec{E}_e^{\alpha,i} = \vec{z}A^- e^{ik_0(x\cos\varphi + y\sin\varphi)}, \quad \vec{E}_e^{\alpha,r} = \vec{z}A^- R_\alpha^{TE} e^{ik_0(x\cos\varphi - y\sin\varphi)} \tag{12}$$

$$\vec{H}_m^{\alpha,i} = -\vec{z}A^+ e^{ik_0(x\cos\varphi + y\sin\varphi)}, \quad \vec{H}_m^{\alpha,r} = -\vec{z}A^+ R_\alpha^{TM} e^{ik_0(x\cos\varphi - y\sin\varphi)} \tag{13}$$

Defining impedances as ratio of tangential components of the fields as $\eta_\alpha^{TE} = E_z / H_x$, $\eta_\alpha^{TM} = -E_x / H_z$, we have

$$\eta_\alpha^{TE} = -\frac{(\eta - \eta^{-1}\tan(\pi\alpha/2)) + (1 - \tan(\pi\alpha/2))\sin\varphi}{(\eta\tan(\pi\alpha/2) - \eta^{-1})\sin\varphi - (1 - \tan(\pi\alpha/2))}, \quad \eta_\alpha^{TM} = \frac{(\eta + \eta^{-1}\tan(\pi\alpha/2))\sin\varphi + (1 + \tan(\pi\alpha/2))}{(\eta\tan(\pi\alpha/2) + \eta^{-1}) + (1 + \tan(\pi\alpha/2))\sin\varphi} \tag{14}$$

Therefore fractional fields satisfy boundary conditions $E_z^\alpha = \eta_\alpha^{TE} H_x^\alpha$, $E_x^\alpha = -\eta_\alpha^{TM} H_z^\alpha$, $y \to +0$.

Two impedances η_α^{TE} and η_α^{TM} are not equal to each other, so these boundary conditions describe IBC for an anisotropic material [3]

$$\vec{n} \times \vec{E}^\alpha = -\bar{\bar{\eta}}_\alpha \cdot \vec{n} \times (\vec{n} \times \eta_0 \vec{H}^\alpha) \tag{15}$$

where $\bar{\bar{\eta}}_\alpha = (\eta_\alpha^{TE}\vec{x})\vec{x} + (\eta_\alpha^{TM}\vec{z})\vec{z}$. If $\alpha = 0$ or $\alpha = 1$ in (15) then we get the usual IBC for an isotropic material with impedance η or η^{-1}, respectively.

Assume that the original impedance $\eta = 0$ indicating a perfectly electric conducting (PEC) surface. Then, analyzing expressions for the fractional fields (9),(10) and new impedance $\bar{\bar{\eta}}_\alpha$, we can state that (i) if $\alpha = 0$, we get the original fields describing the solution for PEC surface ($\eta_\alpha^{TE} = \eta_\alpha^{TM} = \eta = 0$), (ii) if $\alpha = 1$, we obtain the dual fields which are the solution for a perfectly magnetic conducting surface ($\eta_\alpha^{TE} = \eta_\alpha^{TM} = 1/\eta = i\infty$), and (iii) if $0 < \alpha < 1$, we have the fields solving the problem of reflection from an anisotropic surface defined by impedance $\bar{\bar{\eta}}_\alpha$. This can be considered as a generalization of the duality principle for the reflection problems: having a solution for a PEC surface we can obtain the solution for an anisotropic surface with impedance $\bar{\bar{\eta}}_\alpha$ by applying fractional $curl^\alpha$ operator to the original fields.

Using presentations for fractional fields we can derive new boundary conditions from (15) as

$$\vec{n} \times curl^\alpha \vec{E} = -\bar{\bar{\eta}}_\alpha \cdot \vec{n} \times (\vec{n} \times curl^\alpha \eta_0 \vec{H}), \quad y \to +0 \tag{16}$$

characterized by the "fractional impedance" $\bar{\bar{\eta}}_\alpha$ and the fractional order α.

IV. CONCLUSION

In this paper, we have obtained a presentation of the fractional $curl^\alpha$ operator for the plane wave field functions. By using this presentation we have analyzed the fractional fields in classical two-dimensional problem of oblique plane wave incidence on the impedance surface. We have showed that fractional fields represent the solution of reflection problem from an anisotropic surface. We have also derived new reflection coefficients R_α^{TE}, R_α^{TM}.

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