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Fractional operators approach in wave propagation, reflection and radiation problems

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Abstract
We review potential applications of the tools of fractional derivatives and fractional curl operator to propagation, radiation and reflection problems in electromagnetics. New intermediate fractional sources as extensions to canonical sources are analyzed. It is shown that in reflection problems fractional field represents a solution for new type of boundaries, which in special cases can be described by anisotropic or bi-anisotropic boundary conditions. Besides, propagation in bi-isotropic medium can be described by the concept of fractional field.

1. Fractional sources using fractional integrals

Tools of fractional calculus in electromagnetics have been extensively considered in the works of N. Engheta [1-3]. In this section a new definition of the fractional derivatives of function $\psi(\vec{r})$, which satisfies inhomogeneous Helmholtz equation, is given. As will be shown later, we come to such representation after considering "fractional" solutions of ordinary Helmholtz equation.
We use fractional integral known as Riemann-Liouville integral:

$$ {}_{-\infty}D_x^{\alpha} f(x) \equiv \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{x} (x-t)^{-\alpha-1} f(t)dt \text{, for } \alpha < 0, \quad (1)$$

where $\Gamma(-\alpha)$ is the Gamma function. We will use the uniform symbol ${}_{-\infty}D_x^{\alpha}$ (or D_x^{α}) to denote both fractional derivative and fractional integral, and it will define a fractional derivative for $0<\alpha<1$ and a fractional integral for $\alpha<0$.

Green's Theorem with fractional derivatives

Consider a function $\psi(\vec{r})$, which satisfies inhomogeneous scalar Helmholtz equation with the source density given by function $\rho(\vec{r})$:

$$\Delta\psi(\vec{r})+k^2\psi(\vec{r}) = -4\pi\rho(\vec{r}). \quad (2)$$

Besides, define $G(\vec{r},\vec{r}_0)$ as the Green's function of the Helmholtz equation:

$$\Delta G(\vec{r},\vec{r}_0)+k^2 G(\vec{r},\vec{r}_0) = -4\pi\delta(\vec{r}-\vec{r}_0), \quad (3)$$

Here, $\delta(\vec{r}-\vec{r}_0)$ is the three-dimensional Dirac delta function, $\vec{r}$ and $\vec{r}_0$ are the position vectors for the observation and source points, respectively, $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the Laplacian, and k is a scalar constant. After applying fractional derivatives to equations (2) and (3) with respect to the x variable, multiplying the first equation with ${}_{-\infty}D_x^{\nu}G(\vec{r},\vec{r}_0)$, and the second with ${}_{-\infty}D_x^{\mu}\psi(\vec{r})$, subtracting one from another, integrating this over all source coordinates x_0, y_0, z_0 inside S_0, and finally using the Green's theorem, we obtain the following representation:

$$ {}_{-\infty}D_x^{\beta}\psi(\vec{r}) = \begin{cases} \int_V {}_{-\infty}D_{x_0}^{\beta-\nu}\rho(\vec{r}_0)\cdot {}_{-\infty}D_{x_0}^{\nu}G(\vec{r},\vec{r}_0)dv_0 + \\ +\frac{1}{4\pi}\oint_{S_0}[\,{}_{-\infty}D_{x_0}^{\nu}G(\vec{r},\vec{r}_0)\cdot\nabla_0\,{}_{-\infty}D_{x_0}^{\beta-\nu}\psi(\vec{r}_0) - {}_{-\infty}D_{x_0}^{\beta-\nu}\psi(\vec{r}_0)\cdot\nabla_0\,{}_{-\infty}D_{x_0}^{\nu}G(\vec{r},\vec{r}_0)]\cdot ds_0 & \text{, for } \vec{r}\in V \\ 0 & \text{, for } \vec{r}\notin V \end{cases} \quad (4)$$

where $\mu+\nu=\beta$. Here it was used the property of the fractional derivative of the Dirac delta function:

$$\int_{V_0} F(\vec{r}_0)\,{}_{-\infty}D^{\nu}_{x_0}\delta(\vec{r}_0-\vec{r})dv_0 = {}_{-\infty}D^{\nu}_{x}F(\vec{r}), \tag{5}$$

Equation (4) is a generalization of well-known Green's theorem for the case of fractional derivatives. Consider some important particular cases, which can be obtained from (4).

Case 1. Excitation in free space

For the case when volume V is the whole space, the surface integrals in (4) vanish and we have:

$$_{-\infty}D^{\beta}_{x}\psi(\vec{r}) = \int_{V} {}_{-\infty}D^{\beta-\nu}_{x_0}\rho(\vec{r}_0)\bullet {}_{-\infty}D^{\nu}_{x_0}G(\vec{r},\vec{r}_0)dv_0 \tag{6}$$

Originally function $\psi(\vec{r})$ characterizes the field excited by the source with the volume density $\rho(\vec{r})$. From the other hand, for $\beta=0$ representation (6) means that the field $\psi(\vec{r})$ is expressed through the distribution of fractional sources with density $D^{-\nu}\rho(r_0)$ inside volume V and by using fractional integral of conventional Green's function $D^{\nu}G(r_0,r)$.

Case 2. Assuming $\rho(\vec{r})=0$, we can obtain some other important representations:

$$_{-\infty}D^{\beta}_{x}\psi(\vec{r}) = \begin{cases} \frac{1}{4\pi}\oint_{S_0}[\,{}_{-\infty}D^{\beta}_{x_0}G(\vec{r},\vec{r}_0)\bullet\nabla_0\psi(\vec{r}_0) - \psi(\vec{r}_0)\bullet\nabla_0\,{}_{-\infty}D^{\beta}_{x_0}G(\vec{r},\vec{r}_0)]ds_0, & if\ \nu=\beta,\mu=0 \\ \frac{1}{4\pi}\oint_{S_0}[G(\vec{r},\vec{r}_0)\bullet\nabla_0\,{}_{-\infty}D^{\beta}_{x_0}\psi(\vec{r}_0) - {}_{-\infty}D^{\beta}_{x_0}\psi(\vec{r}_0)\bullet\nabla_0 G(\vec{r},\vec{r}_0)]ds_0, & if\ \nu=0 \end{cases} \tag{7}$$

From this representation we see that the fractional derivative of function $\psi(\vec{r})$ is expressed either via the value of the function and its first derivative at the boundary and the fractional derivatives of Green's function, or by the fractional derivatives of the function at the boundary and the usual Green's function.

Case 3. When $\nu=-\mu$, i.e. $\beta=0$ we obtain a representation for the function $\psi(\vec{r})$ itself:

$$\psi(\vec{r}) = \frac{1}{4\pi}\oint_{S_0}[\,{}_{-\infty}D^{-\mu}_{x_0}G(\vec{r},\vec{r}_0)\bullet\nabla_0\,{}_{-\infty}D^{\mu}_{x_0}\psi(\vec{r}_0) - {}_{-\infty}D^{\mu}_{x_0}\psi(\vec{r}_0)\bullet\nabla_0\,{}_{-\infty}D^{-\mu}_{x_0}G(\vec{r},\vec{r}_0)]ds_0 \tag{8}$$

This expression means that the function $\psi(\vec{r})$ is represented through its fractional derivatives at the boundary and the fractional derivatives of Green's function.

Generalization of the Huygens principle

Equations (7), (8) generalize the Huygens principle in such a sense that the fractional derivative of the function $\psi(\vec{r})$, which characterizes a wave process, is presented as a superposition of waves radiated by elementary "fractional" sources distributed on the given surface. "Fractional" potentials, $\oint_{S_0} {}_{-\infty}D^{\beta-\nu}_{x_0}\psi(\vec{r}_0)\bullet\nabla_0\,{}_{-\infty}D^{\nu}_{x_0}G(\vec{r},\vec{r}_0)\bullet ds_0$, $\oint_{S_0} {}_{-\infty}D^{\nu}_{x_0}G(\vec{r},\vec{r}_0)\bullet\nabla_0\,{}_{-\infty}D^{\beta-\nu}_{x_0}\psi(\vec{r}_0)\bullet ds_0$, can be considered as generalization to well-known single and double layer potentials.

Fractional boundary conditions and integral equations

This generalization of Green's theorem can be used to obtain integral equations for various boundary problems. To do this we should complete the equation (7) with appropriate boundary conditions. We introduce the "fractional" boundary condition

$$_{-\infty}D^{\alpha}\psi(\vec{r})|_{S_0} = 0 \tag{9}$$

If $\alpha=0$ or $\alpha=1$, then we have conventional Dirichlet or Neumann boundary condition, respectively (if one chooses the variable of fractional differentiation appropriately). Using equation (7) for $\nu=0$ and taking $\vec{r}\to S_0$, we can obtain a boundary integral equation.

For open regions the radiation conditions at $\vec{r}\to\infty$ should also be considered.

Intermediate sources between line and sheet currents

Having line current distribution $\vec{j}^{\,1}_e(\vec{r}) = \vec{z}\delta(x)\delta(y)$, a new fractional source can be proposed [2]:

$$\vec{j}_e^{\alpha}(\vec{r}) = {}_eS_{\alpha}(x,y) = \vec{z}\frac{1}{2}[{}_{-\infty}D_y^{-\alpha}\delta(x)\delta(y) - {}_{-\infty}D_{-y}^{-\alpha}\delta(x)\delta(y)] = \vec{z}\frac{1}{2}\frac{\delta(x)\,|\,y\,|^{\alpha-1}}{2\Gamma(\alpha)} \quad \text{for } 0<\alpha<1 \qquad (10)$$

For $\alpha = 0$ this source represents a line source: $\vec{j}_e^{1}(\vec{r}) = \vec{z}\delta(x)\delta(y)$. For $\alpha = 1$, $\vec{j}_e^{2}(\vec{r}) = \vec{z}\frac{1}{2}\delta(x)$ is a sheet current source. Fractional derivative of delta function in (10) can be considered as "intermediate" source between one and two-dimensional sources defined by one and two-dimensional delta-functions $\delta(x)$ and $\delta(x)\delta(y)$, respectively.

Fractional Green's function and intermediate waves

Intermediate source corresponds to the fractional Green's function ${}_eG_f(x,y;k)$ as a solution of the Helmholtz equation with the function (10) in the right-hand side. Fractional Green's function defines a new wave [3], which has features of "intermediate" wave between plane and cylindrical waves. Engheta proposed the following form for the fractional Green's function:

$${}_eG_f(x,y;k) = \vec{z}\frac{1}{2}[{}_{-\infty}D_y^{-\nu}G_2(x,y) - {}_{-\infty}D_{-y}^{-\nu}G_2(x,y)] \cong \frac{i}{4\pi}\cos(\frac{\pi\nu}{2})(k\sin|\varphi|)^{-\nu}\sqrt{\frac{2\pi}{k\rho}}e^{ik\rho - i\pi/4} + \frac{i}{4k^{\nu}\Gamma(\nu)}\frac{e^{ik|x|}}{(k\,|\,y\,|)^{1-\nu}}$$

for $k\rho \to \infty$, $\varphi \neq 0$ but φ not too small, $G_2(x,y) = H_0^{(1)}(k\sqrt{x^2+y^2})$ is a two-dimensional Green's function. In the far zone this Green's function has a plane-wave part in addition to the cylindrical wave. It is a nonuniform plane wave propagating in the y direction and its amplitude behaves with y as $|\,y\,|^{\nu-1}$.

2. Fractional sources using fractional curl operator

Fractional curl operator $curl^{\alpha}$ was introduced in [1]. The fractional order α can be, in general, a complex number. We have analyzed radiation from elementary fractional sources such as sheet currents, line currents and point dipoles. In the limit cases of $\alpha = 0$ and $\alpha = 1$ the "fractional" from the given electric field represents the fields due to electric and magnetic currents, respectively. The expressions for the new "fractional" sources were derived from the "fractional" fields. It is shown that the fractional sources represent a coupling of electric and magnetic currents and can be treated as intermediate case between electric and magnetic sources. Such fractional sources were analyzed in works [4,5,6].

The technique for fractionalization of operator was described in [1] where the presentation for $curl^{\alpha}$ of the function of one variable was presented. Following this recipe we get the presentation for $curl^{\alpha}\vec{F}$ where the vector function $\vec{F}$ is expressed via exponents as $\vec{F} = \vec{x}e^{iax+iby+icz}$:

$$\begin{aligned} curl^{\alpha}[\vec{x}e^{iax+iby+icz}] &= \frac{1}{k^2}\{\vec{x}i^{\alpha}(\delta_{0\alpha}a^2 + Bk^{\alpha}(b^2+c^2)) + \\ &+\vec{y}i^{\alpha}(\delta_{0\alpha}ab + k^{\alpha}B(Akc - Bab)) + \vec{z}i^{\alpha}(\delta_{0\alpha}ac - k^{\alpha}B(Akb + Bac))\}e^{iax+iby+icz} \end{aligned}, \qquad (11)$$

where $B = \cos(\frac{\pi\alpha}{2})$, $A = \sin(\frac{\pi\alpha}{2})$.

The fractional field $(\vec{E}^{\alpha}, \vec{H}^{\alpha})$ is defined as application of $curl^{\alpha}$ to the original field $(\vec{E}, \vec{H})$ radiated from some current distribution (electric current with density $\vec{j}_e$ and magnetic current with $\vec{j}_m$):

$$\vec{E}^{\alpha} = (ik_0)^{-\alpha}curl^{\alpha}\vec{E}; \quad \eta_0\vec{H}^{\alpha} = (ik_0)^{-\alpha}curl^{\alpha}(\eta_0\vec{H}), \qquad (12)$$

where $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$ is the propagation constant, $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the impedance, μ_0, ε_0 are the permittivity and permeability of the medium.

Fractional field is a field radiated by new sources $(\vec{j}^{e,\alpha}, \vec{j}^{m,\alpha})$ in the medium with (ε_0, μ_0). We name these currents "fractional" or "intermediate" currents. Field (12) can be considered as "intermediate" or "fractional" solution between the original field and the dual field.

It can be shown that the currents $(\vec{j}^{e,\alpha},\vec{j}^{m,\alpha})$ are a combination of the original currents $(\vec{j}^{e},\vec{j}^{m})$ [5]:

$$\vec{j}_e^{\alpha} = \cos(\frac{\pi\alpha}{2})\vec{j}_e + \sin(\frac{\pi\alpha}{2})\vec{j}_m, \quad \vec{j}_m^{\alpha} = -\sin(\frac{\pi\alpha}{2})\vec{j}_e + \cos(\frac{\pi\alpha}{2})\vec{j}_m. \quad (13)$$

Note that fractional currents are distributed in the same volume as original currents. Fractional source (13) can be treated as a coupling of original magnetic and electric sources, i.e. $curl^{\alpha}$ makes a mixture of electric and magnetic properties. Besides, fractional electric current can be treated as intermediate source between electric and magnetic currents. Next we consider some important examples of excitations in free space: radiation from sheet currents, line currents and point dipoles.

Fractional Sheet Currents

Consider the field $(\vec{E},\vec{H})$ radiated from a combination of two parallel sheets of currents located in the plane $y=0$: a sheet of electric current $\vec{j}_e = \vec{x}j_e$ and a sheet of magnetic current - $\vec{j}_m = \vec{x}j_m$, where

$$\vec{j}_e = \vec{x}J_e e^{i\beta_0 x}\delta(y), \quad \vec{j}_m = -\eta\vec{x}J_m e^{i\beta_0 x}\delta(y), \quad (14)$$

and $J_e = |J_e| e^{-i\psi_e}$, $J_m = |J_m| e^{-i\psi_m}$ are the amplitudes, ψ_0 is the initial phase, β_0 is the propagation coefficient. Fractional field can be considered as a field excited by a combination of two sheets of electric and magnetic currents $j_e^{\alpha} = \vec{x}J_e^{\alpha}e^{i\beta_0 x}\delta(y)$, $j_m^{\alpha} = -\eta\vec{x}J_m^{\alpha}e^{i\beta_0 x}\delta(y)$, where

$$J_e^{\alpha} = |J_e^{\alpha}| e^{i\psi_e^{\alpha}} = BJ_e - AJ_m, \quad J_m^{\alpha} = |J_m^{\alpha}| e^{i\psi_m^{\alpha}} = AJ_e + BJ_m. \quad (15)$$

Polarization of the fractional field

Polarization of the fractional field depends on polarization of the original field and the fractional order α. We consider special case when $\beta_0 = 0$, and also we assume, that $\psi_e = 0$.

The polarization vector for the electric field $\vec{E}^{\alpha}$ can be expressed as

$$P^{\alpha}(t) = Re\{(2|E^{\alpha}|)^{-1}\eta[\vec{x}(BJ_e - AJ_m) + \vec{z}(\mp AJ_e \mp BJ_m)]\cos(-\omega t \pm ky)\} \quad (16)$$

Consider two special cases of the original field: linearly-polarized and elliptically-polarized.

Case 1. The original field is linearly-polarized. If α is real then the electric fractional field is linearly-polarized and the polarization vector makes the angle δ_{α} with the axis x defined by

$$\tan\delta_{\alpha} = \mp|J_e^{\alpha}|/|J_m^{\alpha}|. \quad (17)$$

If α is complex then the fractional field is elliptically polarized.

Case 2. The original field is elliptically-polarized. If elliptically-polarized original field is characterized by the ellipse with axes $a = J_e$, $b = J_m$, then the fractional field for $0 < \alpha < 1$ is elliptically-polarized but with the axes rotated by some angle to coordinate axes. For $\alpha = 1$ this ellipse has horizontal axis $a_{\alpha} = b$ and vertical axis $b_{\alpha} = a$. For the case when the original field is circularly polarized, the fractional field remains circularly polarized.

Fractional line currents

Consider a field $(\vec{E},\vec{H})$ radiated by the line current located along the axis z. The current density is given as $\vec{j}_e = \vec{z}J_e\delta(x)\delta(y)$. It can be shown that the fractional field from such original field is a field due to a combination of two line currents, electric and magnetic ones, with the densities

$$\vec{j}_e^{\alpha} = \vec{z}J_e\cos(\frac{\pi\alpha}{2})\delta(x)\delta(y), \quad \vec{j}_m^{\alpha} = \vec{z}J_e\sin(\frac{\pi\alpha}{2})\delta(x)\delta(y). \quad (18)$$

For $\alpha = 1$ the fractional field is a field radiated only by a magnetic line current, $\vec{j}_m^{\alpha}|_{\alpha=1} = \vec{z}J_e\delta(x)\delta(y)$.

Fractional point dipoles

Consider a field $(\vec{E}^e,\vec{H}^e)$ radiated by an ideal electric Hertz dipole [1] located at the origin and having the dipole moment $\vec{p} = p\vec{x}$, where $p = const$. The electric current is distributed with the density

$\vec{j}_e(\vec{r}) = \vec{p}\delta(r)$. The fractional field from this field can be treated as a field due to the fractional sources $(\vec{j}_{e,\alpha}, \vec{j}_{m,\alpha})$, which are a combination of the electric and magnetic dipoles:

$$\vec{j}_{e,\alpha} = B\vec{j}_e\,,\ \vec{j}_{m,\alpha} = -A\vec{j}_e \qquad (19)$$

From the other hand, the field $\vec{E}^{\alpha,e}$ should satisfy the Helmholtz equation with the function $\mathrm{curl}^\alpha(\vec{j}_e)$ in the right-hand side, i.e. the fractional field $(\vec{E}^{\alpha,e}, \vec{H}^{\alpha,e})$ is the field due to the electric current

$$\vec{j}^{\alpha,e} = -\mathrm{curl}^\alpha(\vec{j}_e) \qquad (20)$$

Fractional electric current $\vec{j}^{\alpha,e}$ (20) and fractional sources (19) are equivalent it the sense that these two source distributions radiate the same field.

Physical remarks about fractional sources

Fractional sources obtained by applying fractional operators have the following physical meaning:

(1) Fractional curl operator yields fractional sources, which are effectively an intermediate case between electric and magnetic sources.

(2) Fractional integral yields the change of source density (change in the volume where the source is distributed) while not changing the nature of the source (electric source remains electric, etc.).

3. Fractional curl operator in reflection problems

In this section, new boundaries obtained by the fractional operators approach are considered. Using the operator $curl^\alpha$, Engheta [1] obtained new boundary with impedance $\eta_\alpha = itg(\pi\alpha/2)$ as intermediate case between perfect electric conductor (PEC) and perfect magnetic conductor (PMC) in the problem of normal incidence of a plane wave on an impedance boundary. Lindell and Sihvola [7] considered new boundary called Perfectly Electromagnetic Conducting Boundary (PEMC) as a generalization of PEC and PMC defined by boundary conditions (BC) like $\vec{E} + M\vec{H} = 0$ with the coefficient M named the admittance.

Consider a classic two-dimensional problem of the TE polarized plane wave $\vec{E}^i(0,0,e^{ik(x\cos\varphi + y\sin\varphi)})$ incidence at the angle φ on the plane boundary, $y = 0$. The total field $(\vec{E}, \vec{H})$ should satisfy isotropic impedance BC with the impedance η [8]. Having the field $(\vec{E}, \vec{H})$ as a known solution for a certain value of impedance η, we can construct the fractional field as $curl^\alpha$ applied to this field $(\vec{E}.\vec{H})$. Fractional field $(\vec{E}^\alpha, \vec{H}^\alpha)$ acts as an intermediate solution between the original and dual solutions [1]. Duality principle in the reflection problems states that the dual solution $(\vec{E}^1, \vec{H}^1)$, obtained as $\vec{E}^1 = \eta_0\vec{H}, \eta_0\vec{H}^1 = -\vec{E}$, corresponds to an impedance surface with η^{-1} when the original solution is a solution for the impedance η. As a special case, the dual field corresponds to the PMC boundary when the original field is a solution for the PEC one.

We assume that the fractional field represents the solution of the reflection problem for some boundary, which can be characterized by the following BC [8]:

$$\vec{n} \times \vec{E}^\alpha = \overline{\eta}_\alpha \vec{n} \times (\vec{n} \times \vec{H}^\alpha), \qquad (21)$$

where the impedance $\overline{\eta}_\alpha$, in general, can be written as a tensor $\overline{\eta}_\alpha = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ and can be expressed via the original impedance η and the order α.

Consider two important special cases of $\overline{\eta}_\alpha$.

Case 1. Anisotropic BC ($t_{12} = t_{21} = 0$). It can be shown that [9,10]

$$t_{11} = i\frac{1}{\sin\varphi}\tan(\pi\alpha/2)\frac{1 - i\eta\sin\varphi\cot(\pi\alpha/2)}{1 + i\eta\sin\varphi\tan(\pi\alpha/2)},\ t_{22} = i\sin\varphi\tan(\pi\alpha/2)\frac{1 - i\eta\sin\varphi\cot(\pi\alpha/2)}{1 + i\eta\sin\varphi\tan(\pi\alpha/2)} \qquad (22)$$

If the original field is a solution for a PEC ($\eta = 0$) boundary, then we have

$$t_{11} = i\frac{1}{\sin\varphi}\tan(\pi\alpha/2),\ t_{22} = i\sin\varphi\tan(\pi\alpha/2) \qquad (23)$$

For the normal incidence ($\varphi = \pi/2$) from (23): $t_{11} = t_{22} = i\tan(\pi\alpha/2)$. Similar formula was derived by Engheta in [1] for the normal incidence on isotropic impedance boundary, and the expressions (23) are the extension to the case of oblique incidence of the plane wave.

Case 2. Bi-anisotropic BC ($t_{11} = t_{22} = 0$). In this case the coefficients are expressed as [9,10]

$$t_{12} = -i\frac{1 - i\eta\sin\varphi\cot(\pi\alpha/2)}{1 + i\eta\sin\varphi\tan(\pi\alpha/2)},\quad t_{21} = -i\tan^2(\pi\alpha/2)\frac{1 - i\eta\sin\varphi\cot(\pi\alpha/2)}{1 + i\eta\sin\varphi\tan(\pi\alpha/2)} \tag{24}$$

If the original field is a solution for a PEC ($\eta = 0$) boundary then $t_{12} = -i$, $t_{21} = -i\tan^2(\pi\alpha/2)$. For $0 < \alpha < 1$, (24) describes a bi-anisotropic boundary, which supports surface currents given by

$$\vec{j}_e^{\,\alpha} = -\bar{\eta}_\alpha \vec{j}_m^{\,\alpha} \tag{25}$$

4. Fractional field in a medium

We propose tool of fractional curl operator to describe fields propagating in the bi-isotropic medium. The constitutive relations for bi-isotropic media can be written in the form

$$\vec{D} = \varepsilon\vec{E} + \xi\vec{H},\ \vec{B} = \varsigma\vec{E} + \mu\vec{H} \tag{26}$$

where $(\varepsilon, \mu, \xi, \varsigma)$ are scalars: $\xi = (\chi - ik)\sqrt{\varepsilon_0\mu_0}$, $\varsigma = (\chi + ik)\sqrt{\varepsilon_0\mu_0}$, which are the Tellegen and chirality parameters, respectively. We will consider the case of a purely chiral medium with $\chi = 0$.

After applying rot^α to the known field (E,H), which represents propagating wave in homogeneous isotropic medium, we obtain new fractional field $(\vec{E}^\alpha, \vec{H}^\alpha)$:

$$(\vec{D}^\alpha, \vec{B}^\alpha) = (ik)^{-\alpha}\, curl^\alpha(\vec{D}, \vec{B}). \tag{27}$$

Fractional field (27) can be treated as a field radiated from the same sources and propagating in a new medium characterized by parameters expressed via the fractional order α and original parameters (ε_0, μ_0). For the case when original field is a propagating plane wave, the fractional field has simple analytical form:

$$\vec{E}^\alpha = B\varepsilon_0\vec{E}^0 + A\mu_0\vec{H}^0,\ H^\alpha = B\mu_0\vec{H}^0 - A\varepsilon_0\vec{E}^0 \tag{28}$$

Comparing field (26) in bi-isotropic medium and fractional field (28) we can conclude that fractional field represents a field in the bi-isotropic media with parameters $(\varepsilon, \mu, \xi, \varsigma)$ expressed as:

$$\varepsilon = \varepsilon_0\cos(\pi\alpha/2),\ \mu = \mu_0\cos(\pi\alpha/2),\ \xi = \mu_0\sin(\pi\alpha/2),\ \varsigma = -\varepsilon_0\sin(\pi\alpha/2) \tag{29}$$

Operator curl^α can be treated as the operator that connects the fields in the usual simple isotropic medium and the complex bi-isotropic medium.

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