Effective method of solution of diffraction problems for waves incident on a cylindrical screen

V. N. Vavilov, É. I. Veliev, V. V. Veremeĭ, and V. P. Shestopalov[1)]

Institute of Radiophysics and Electronics, Academy of Sciences of the Ukrainian SSR, Khar'khov

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A rigorous solution of the diffraction problem for waves incident on a circular cylindrical screen can be obtained with the help of the Riemann–Hilbert method.[1] In this method dual series equations with trigonometric function Kernels are constructed for constructed for the Fourier coefficients of the current density on the screen. They can be reduced to an infinite system of linear algebraic equations of the second kind by the semi-inversion method. However, the Fourier coefficients fall off only as $m^{-\alpha}$ $(1/2 \le \alpha \le 3/2)$ for large values of the summation index m and therefore the calculation of the current density function is complicated because it is necessary to sum a slowly converging series.

In the present paper we apply systematically the method of moments[2] and then the semi-inversion procedure to solve the dual series equations and obtain a system of linear algebraic equations in which the unknowns are the coefficients of the expansion of the surface current density function in the complete orthogonal system of Gegenbauer polynomials with a weight function that takes into account the behavior of the current density function on the edges of the screen.

The effectiveness of our approach in comparison with the known results following from the Riemann–Hilbert method[1] is demonstrated by numerical calculations of the current density on the screen and the angular dependence of the scattered field.

1. Let an arbitrary E-polarized electromagnetic field $E_z^0 = f(r, \phi)$ excite a cylindrical screen of radius a with angular width 2θ and orientation angle ψ_0 (see Fig. 1). The time dependence of the

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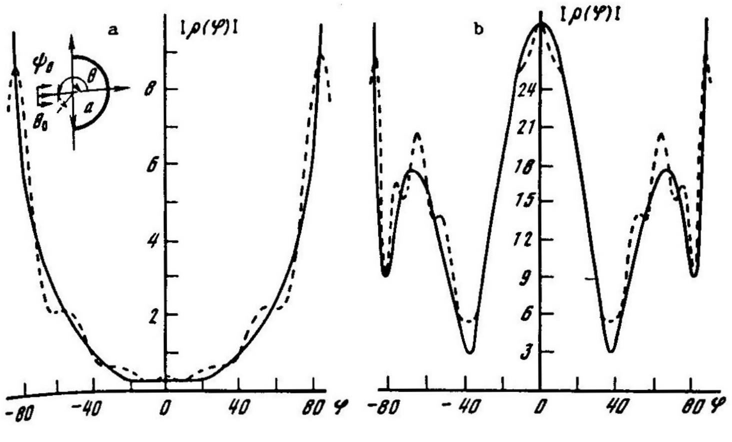

FIG. 1

form $e^{-i\omega t}$ is omitted. It is required to find the electromagnetic field arising as a result of scattering of the wave E_z^0 by the cylinder. This field satisfies the Helmholtz equation outside the surface of the cylinder L, the Sommerfeld radiation condition at infinity, a Dirichlet boundary condition at the surface of the cylinder, and the condition that the energy within any bounded volume of space be finite. The single nonzero component of the scattered electric field is written in the form [1]

$$E_z^s(r,\varphi)=\sum_{m=-\infty}^{\infty}\rho_m\begin{Bmatrix}J_m(ka)H_m^{(1)}(kr)\\ J_m(kr)H_m^{(1)}(ka)\end{Bmatrix}e^{im\varphi},\quad \begin{matrix}r>a,\\ r<a,\end{matrix} \tag{1}$$

where $k = 2\pi/\lambda$, λ is the wavelength, $J_m(x)$ and $H_m^{(1)}(x)$ are Bessel and Hankel functions, respectively; and $\{\rho_m\}_{m=-\infty}^{\infty}$ are the Fourier coefficients of the function $\rho(\phi)$, which is defined to be zero outside the angular interval of the metallic screen and inside this interval it is equal to the surface current density on the screen

$$\rho(\varphi)=\begin{cases}\dfrac{2}{i\pi a}\displaystyle\sum_{m=-\infty}^{\infty}\rho_m e^{im\varphi}, & \varphi\in[-\theta,\theta],\\ 0, & \varphi\notin[-\theta,\theta].\end{cases} \tag{2}$$

Representing the right hand side of the Helmholtz equation $f(r, \phi)$ as a Fourier series in the coordinates (r, ϕ) and applying the Dirichlet boundary condition for the total field $E_z = (E_z^0 + E_z^S)|_L \equiv 0$, we obtain a system of dual series equations with trigonometric functions for the kernel [1]

$$\begin{aligned}&\sum_{m=-\infty}^{\infty} y_m\gamma_m e^{im\varphi}=-\sum_{m=-\infty}^{\infty}F_m e^{im\varphi},\quad \varphi\in[-\theta,\theta],\\ &\sum_{m=-\infty}^{\infty} y_m e^{im\varphi}=0,\quad \varphi\in[\theta,2\pi-\theta],\end{aligned} \tag{3}$$

where $y_m=\rho_m(-1)^m e^{im\psi_0}$, $F_m=f_m(-1)^m e^{im\psi_0}$, and $\gamma_m=J_m(ka)H_m^{(1)}(ka)$. The unknown coefficients ρ_m belong to the space of number sequences $\tilde{\ell}_2$, where

$$\tilde{l}_2=\{\rho_m:\sum_{(m)}|\rho_m|^2(|m|+1)^{-1}\}<\infty.$$

The quantity γ_m can be written in the form

$$\gamma_m=\frac{1}{i\pi|m|}(1-\delta_m),\quad m\neq 0;\quad \delta_m=\delta_m\underset{m\to\infty}{\sim}O\left(\frac{1}{m^2}\right);$$
$$\delta_m=1-i\pi|m|J_m(ka)H_m^{(1)}(ka),$$

which corresponds to decomposing the operator of the problem into its principal and continuous parts.

Therefore, the system (3) reduces to the form

$$\begin{aligned}&\sum_{(m\neq 0)}\frac{y_m}{|m|}(1-\mathcal{E}_m)e^{im\theta\eta}+y_0\gamma_0=-i\pi\sum_{(m)}F_m e^{im\theta\eta},\quad |\eta|<1,\\ &\sum_{m=-\infty}^{\infty}y_m e^{im\theta\eta}=0,\quad |\eta|>1,\end{aligned} \tag{4}$$

where $\eta = \phi/\theta$, and $\rho(\eta) \equiv \rho(\phi)$.

The system of equations (4) was studied earlier [1] assuming that the current density function near the edges of the screen has a singularity of the form $\rho(r)\underset{r\to 0}{\sim}O(r^{\nu-1})$, with $\nu = \frac{1}{2}$, which corresponds to an infinitely thin screen. Below we give a more general solution of (4) in the case where the parameter ν, which describes the singularity of the function $\rho(\eta)$, can have any value in the interval $[\frac{1}{2}, 1[$; hence our solution contains the earlier obtained results as a special case. This generalization is important in order to use the method to solve diffraction problems for waves incident on cylindrical bodies formed from parts of intersecting circular cylinders (screens of finite thickness).[3]

2. To construct the solution of system (4), we represent the function $\rho(\eta)$ as a uniformly convergent series of orthogonal Gegenbauer polynomials $\{C_n^{\nu-1/2}(\eta)\}_{n=0}^{\infty}$ with a weight function which takes into account the behavior of the function $\rho(\eta)$ at the ends of the interval $\eta\in[-1, 1]$ (at the edges of the screen)

$$\rho(\eta)=(1-\eta^2)^{\nu-1}\sum_{n=0}^{\infty}x_n C_n^{\nu-1/2}(\eta),\quad ½\leq\nu<1. \tag{5}$$

Using (5), we obtain a new representation for the Fourier coefficients ρ_m of the current density function $\rho(\eta)$:

$$\rho_m=\frac{(-1)^m e^{-im\psi_0\theta}}{\Gamma(\nu-½)}\sum_{n=0}^{\infty}(-i)^n X_n\beta_n^{(\nu-1/2)}\frac{J_{n+\nu-1/2}(m\theta)}{(2m\theta)^{\nu-1/2}},\quad ½<\nu<1, \tag{6}$$

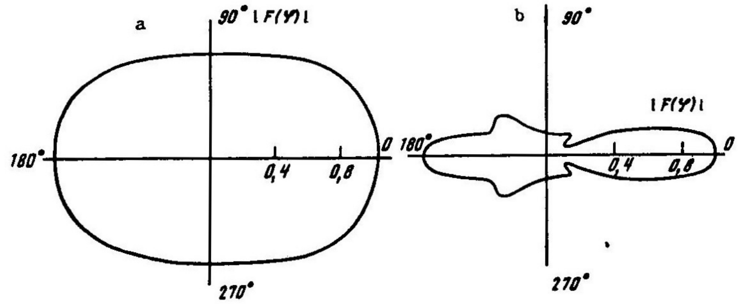

FIG. 2

where $\beta_n^{(\nu-1/2)} = \Gamma(n+2\nu-1)/\Gamma(n+1) \underset{n\to\infty}{\sim} O(n^{2\nu-2})$, $J_{n+\nu-1/2}(x)$ is a Bessel function and $\Gamma(x)$ is the gamma function.

It follows from (6) that $\rho_m \sim O(m^{-\nu})$ in the limit $m \to \infty$ and therefore $\rho_m \in \tilde{l}_2$. If we put $\beta_0{}^{(\nu-1/2)}|_{\nu=1/2} = 1/2$, then this representation will also be valid for $\nu = 1/2$.

To determine the unknowns $\{x_n\}_{n=0}^{\infty}$, we substitute (6) into (4). Using the completeness and orthogonality of the Gegenbauer polynomials, we obtain an infinite system of algebraic equations of the form

$$\sum_{n=0}^{\infty} (-i)^n x_n \beta_n^{(\nu-1/2)} [Q_{k,n}^{(\nu-1/2)} - L_{k,n}^{(\nu-1/2)}] = \Gamma_k, \quad k = 0, 1, 2, \ldots \tag{7}$$

Here we have introduced the notation

$$Q_{k,n}^{(\nu-1/2)} \equiv [1+(-1)^{k+n}] \sum_{m=1}^{\infty} \frac{1}{m} \frac{J_{n+\nu-1/2}(m\theta)}{(m\theta/2)^{\nu-1/2}} \frac{J_{k+\nu-1/2}(m\theta)}{(m\theta/2)^{\nu-1/2}},$$

$$L_{k,n}^{(\nu-1/2)} \equiv (1+(-1)^{k+n}) \sum_{m=1}^{\infty} \frac{\delta_m}{m} \frac{J_{n+\nu-1/2}(m\theta)}{(m\theta/2)^{\nu-1/2}} \frac{J_{k+\nu-1/2}(m\theta)}{(m\theta/2)^{\nu-1/2}} - \frac{i\pi\gamma_0\delta_{k0}\delta_{n0}}{\Gamma^2(\nu+\frac{1}{2})},$$

$$\Gamma_k \equiv \frac{(-i\pi)\cdot 2^{2\nu-1}\Gamma(\nu-\frac{1}{2})}{\theta} \sum_{m=-\infty}^{\infty} F_m \frac{J_{k+\nu-1/2}(m\theta)}{(m\theta/2)^{\nu-1/2}}, \quad \delta_{k0} = \begin{cases} 1, & k=0, \\ 0, & k\neq 0. \end{cases} \tag{8}$$

For the matrix elements $\{Q_{k,n}^{(\nu-1/2)}\}_{k,n=0}^{\infty}$ the series in m can be summed explicitly, while for the matrix elements $\{L_{k,n}^{(\nu-1/2)}\}_{k,n=0}^{\infty}$ the terms in the series in m have the asymptotic form $\sim O(m^{-(2\nu+3)})$.

It can be shown that the Fredholm alternative is valid for (7). The proof is based on the fact that the norms in Hilbert space ℓ_2 of the matrix operators $Q^{(\nu-1/2)}$ and $L^{(\nu-1/2)}$, which correspond to the matrices $\{Q_{k,n}^{(\nu-1/2)}\}_{k,n=0}^{\infty}$ and $\{L_{k,n}^{(\nu-1/2)}\}_{k,n=0}^{\infty}$, are bounded, and the fact that the operator $Q^{(\nu-1/2)}$ is positive definite. In other words, in the space ℓ_2 the matrix operator $L^{(\nu-1/2)}$ is continuous and the operator $Q^{(\nu-1/2)}$ has a two-sided continuous inverse. It can also be shown that the number sequence $\{\Gamma_k\}_{k=0}^{\infty} \in l_2$. Therefore, the system of linear algebraic equations (7) belongs to the class of operator equations for which the Fredholm alternative[4,8] is valid and it can be solved approximately to any desired level of accuracy with the help of the method of reduction.

In the numerical implementation of our approach one of the important problems was the summation of the infinite series (8) for the matrix elements $L_{k,n}{}^{(\nu-1/2)}$. A computer program for the BESM-6 computer was developed in ALGOL-GDR. The results calculated with this program are described below.

As a special case, we consider a plane electromagnetic wave incident at an angle ϑ_0 from the direction $y < 0$ on an infinitely thin ($\nu = 1/2$), cylindrical screen. Then $E_z^0 = e^{ik(\alpha_0 x+\beta_0 y)}$, where $\alpha_0 = \cos\vartheta_0$, $\beta_0 = \sin\vartheta_0$ and $f_m = J_m(ka)$. The coefficients $\{x_n\}_{n=0}^{\infty}$ were found from (7), then (5) was used to calculate the absolute value of the current density $|\rho(\eta)|$. The angular dependence of the scattered field was calculated from the equation

$$|F(\varphi)| = \left| \sum_{m=-\infty}^{\infty} \rho_m J_m(ka) e^{im\varphi} \right|,$$

where $\{\rho_m\}_{m=-\infty}^{\infty}$ are determined from (6).

The calculated results were compared with those obtained using the Riemann–Hilbert method. The angular dependence (polar plot) of the scattered field is shown in Fig. 2 for two values of the frequency parameter ka [ka = 1 (a) and ka = 2π (b)]. Differences in the polar plots obtained by the two methods are indistinguishable on the graphs; the largest deviation between the numerical values was less then 1.5%.

The current density distribution on the screen is shown in Fig. 1 for the same values of ka in the case of a normally incident wave ($\vartheta_0 = 0$), angular half-width of the metallic screen $\theta = 90°$, and orientation angle of the cylinder $\psi_0 = 180°$. The dashed curves were obtained by the Riemann–Hilbert method and the solid curves were obtained using our method For ka = 1 the largest deviation between the two methods is ~25%, while for ka = 2π the deviation reaches 40%. The execution times (for one parameter ka) for the calculation of the current density function $\rho(\eta)$ are roughly the same for the two methods.

The differences in the results can be explained as follows. The coefficients of the series (2) behave as $\rho_m \sim O(m^{-1/2})$ in the limit $m \to \infty$ and to obtain a satisfactory result, a large number of terms must be summed, which is very difficult because a system of higher-order equations must be solved. The calculation of the current density $\rho(\eta)$ using (5) is much more effective because of the rapid convergence of the series, and because of the presence of the weight function, which takes into account the behavior of the function $\rho(\eta)$ at the ends of the interval [near the edges of the screen ($\eta = \pm 1$) the series (2) cannot be used to calculate the function $\rho(\eta)$].

Our results for ka = 2 π closely correspond to those of Ref. 7, where the convergence of the series (2) was improved (see also Refs. 5 and 6).

It has been shown that our method is effective for the solution of diffraction problems for waves incident on cylindrical screens and can be used to accurately determine the surface current density function without the use of additional procedures to improve the convergence of the series.

1)Academician, Academy of Sciences of the Ukrainian SSR.

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