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FRACTIONAL OPERATORS APPROACH IN ELECTROMAGNETIC WAVE REFLECTION PROBLEMS

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Abstract—Applications of fractional operators approach to electromagnetic waves reflection problems are considered. Reflection properties of fractional or intermediate solutions are analyzed. It is shown that this approach is a useful technique for the description of solutions to the reflection problems for some known media in terms of the fractional order and also obtaining boundaries with new features. In this paper intermediate boundaries are modeled with BI slab with a PEC backing and also using anisotropic impedance boundary conditions. The twist-polarizer effect (crosspolarizing reflector) is described in terms of the specific value of the fractional order.

1. INTRODUCTION

Tools of fractional calculus have found many applications in various problems of electromagnetics. Fractional operators are defined as fractionalizations of some commonly used operators. In this paper we consider fractional curl operator $curl^\alpha$ with a fractional order α, proposed in [1] as fractionalization of the conventional curl operator. The order α can be real, $0 < \alpha < 1$, however, complex values of α can be also considered. The $curl^\alpha$ was introduced in [1] to fractionalize the duality principle in electromagnetic theory. A new electromagnetic

field $(\vec{E}^\alpha, \eta_0\vec{H}^\alpha)$ is defined by applying $curl^\alpha$ to some known field $(\vec{E}^0, \eta_0\vec{H}^0)$, which is a solution of some electromagnetic problem with certain values of input parameters:

$$\vec{E}^\alpha = (\imath k_0)^{-\alpha} curl^\alpha \vec{E}^0$$
$$\eta_0\vec{H}^\alpha = (\imath k_0)^{-\alpha} curl^\alpha(\eta_0\vec{H}^0) \quad (1)$$

Here $k_0 = \omega\sqrt{\epsilon_0\mu_0}$ is the wave number in the medium with permittivity ϵ_0 and permeability μ_0, $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic impedance of the medium. Time dependence is assumed to be $e^{-\imath\omega t}$. The field $(\vec{E}^0, \eta_0\vec{H}^0)$ is referred to as an "original" solution. We name the field $(\vec{E}^\alpha, \eta_0\vec{H}^\alpha)$ as "fractional" or "intermediate" solution between the original and dual fields. If the original field $(\vec{E}^0, \eta_0\vec{H}^0)$ is a given solution of Maxwell's equations, then the fractional field $(\vec{E}^\alpha, \eta_0\vec{H}^\alpha)$ defined by expressions (1) represents another solution of Maxwell's equations with the same values of parameters ϵ_0, μ_0 [1]. For $\alpha = 0$ we obtain the original field $(\vec{E}^0, \eta_0\vec{H}^0)$, and for $\alpha = 1$ the fractional field corresponds to the dual solution $(\eta_0\vec{H}^0, -\vec{E}^0)$.

If the original field is excited by some distribution of source currents then the fractional field obtained from this field corresponds to new "fractional" sources, which generalize canonical sources such as point dipoles and line sources.

Fractional dual solutions and corresponding sources were discussed in [2–7]. Fractional field was analyzed by many authors in various electromagnetic problems: propagation in chiral media [8–10], waveguides [11], reflection problems with boundaries of impedance type [12–14], and with surfaces with fractal properties [15].

We have applied the concept of the fractional operators to two-dimensional (2D) problems of reflection from a media interface (the medium can be modeled by the impedance boundary conditions or can be a slab of some anisotropic or bi-anisotropic material, etc). Our interest to fractional operators in reflection problems is connected with the possibility of simple description of known boundaries as intermediate between canonical ones, and also obtaining the boundaries with new features.

Simple mathematical description of reflection properties of boundaries in electromagnetic theory is a common task in the modeling of scattering problems. Classic example of commonly used boundary conditions (BC) are those of Leontovich type [16, 17]

$$\vec{n} \times \vec{E} = \eta\vec{n} \times (\vec{n} \times \vec{H}), \quad (2)$$

where η is a surface impedance, $\vec{n}$ is the normal unit vector to the boundary. Leontovich BC actually describe intermediate boundaries between Perfect Electric Conductor (PEC) and Perfect Magnetic Conductor (PMC) as special cases of impedance $\eta = 0$, $\eta = \infty$, respectively.

Lindell and Sihvola [18] introduced new boundary called Perfectly Electromagnetic Conducting Boundary (PEMC) as a generalization of PEC and PMC and defined by the boundary conditions (BC) with admittance M

$$\vec{H} + M\vec{E} = 0 \tag{3}$$

For $M = 0$ it is PMC boundary, for $M = \pm\infty$ it describes PEC. Realization of PEMC was given in [19, 20].

Fractional operators approach (FOA) in electromagnetic problems means utilization of fractional operators in description of solutions which are effectively intermediate between canonical solutions of the considered problem. In this paper FOA is used in reflection problems; it is based on introducing new fractional field as the result of application of $curl^\alpha$ to the known "original" solution. Original solution takes into account the main features of solution of original reflection problem. Effect of the fractional order (FO) α yields a coupling of electric and magnetic fields that means changing the polarization of the original field. Fractional field can be considered as a solution of the problem with the same geometry but with new boundary. Properties of this fractional boundary will be defined by FO α and the original boundary. It is evident that if the original solution is a solution for a PEC (PMC) boundary then the fractional solution describes the solution for the boundaries, which generalize PEC and PMC.

Earlier, using $curl^\alpha$, Engheta obtained fractional boundary characterized by the isotropic impedance $\eta_\alpha = \imath\eta_0 \tan(\pi\alpha/2)$ [1], as an intermediate case between PEC and PMC for the problem of normal incidence of a plane wave on a boundary. Generalization to this formula was given in [14] in terms of anisotropic impedance.

As will be shown later, application of $curl^\alpha$ to electromagnetic field results in the coupling of electric and magnetic fields, when, for example, fractional electric field is a combination of original electric and magnetic fields. The same effect is observed in bi-isotropic (BI) media [21]. Therefore we consider the model of the fractional boundary using BI slab.

Proposed FOA can be a useful technique in the description of solutions to reflection problems for some known media in terms of FO α; besides, new boundaries with new features can be obtained. Reflection properties of the fractional solution will be analyzed. In this paper intermediate boundaries are modeled with BI slab with a PEC backing and also using anisotropic impedance boundary conditions.

The twist-polarizer effect (crosspolarizing reflector) can be described in terms of specific value of FO α.

2. PROPERTIES OF THE FRACTIONAL FIELD

The following key feature of the fractional field can be shown: if the original field is a field radiated by the "original" electric and magnetic currents with volume densities $\vec{j}_{e0}$ and $\vec{j}_{m0}$, respectively, then the fractional field represents the field radiated by new electric and magnetic sources, $\vec{j}_{e,\alpha}$ and $\vec{j}_{m,\alpha}$, found from the original ones and FO α [6]:

$$\begin{aligned} \vec{j}_{e,\alpha} &= \cos\left(\frac{\pi\alpha}{2}\right)\vec{j}_{e0} + \sin\left(\frac{\pi\alpha}{2}\right)\vec{j}_{m0} \\ \vec{j}_{m,\alpha} &= -\sin\left(\frac{\pi\alpha}{2}\right)\vec{j}_{e0} + \cos\left(\frac{\pi\alpha}{2}\right)\vec{j}_{m0} \end{aligned} \tag{4}$$

The currents $(\vec{j}_{e,\alpha}, \vec{j}_{m,\alpha})$ are named "fractional currents" referring to the currents, which correspond to the fractional field. Note that fractional currents are distributed within the same volume as the original ones.

To understand the key features of the fractional field, we consider a simple case when the original field is a uniform plane wave propagating along the direction given by the vector $\vec{l}(\cos\phi, \sin\phi, 0)$. This field can be obtained from two independent components,

$$E_z = D_e e^{\imath k(x\cos\phi + y\sin\phi)},\ H_z = D_m e^{\imath k(x\cos\phi + y\sin\phi)}, \tag{5}$$

where D_e, D_m are the amplitudes. Other components are derived from these ones by using Maxwell's equations.

The components of the fractional field $\vec{E^\alpha}(E_x^\alpha, E_y^\alpha, E_z^\alpha)$, $\eta_0\vec{H^\alpha}(\eta_0 H_x^\alpha, \eta_0 H_y^\alpha, \eta_0 H_z^\alpha)$ are expressed via the components of the original field [13, 14]:

$$\begin{aligned} E_z^\alpha &= BE_z + A\eta_0 H_z \\ \eta_0 H_z^\alpha &= -AE_z + B\eta_0 H_z \end{aligned} \tag{6}$$

and other components

$$\begin{aligned} E_x^\alpha &= BE_x + A\eta_0 H_x, E_y^\alpha = BE_y + A\eta_0 H_y \\ \eta_0 H_x^\alpha &= B\eta_0 H_x - AE_x, \eta_0 H_y^\alpha = B\eta_0 H_y - AE_y \end{aligned} \tag{7}$$

Here we denote

$$A = \sin\left(\frac{\pi\alpha}{2}\right), \quad B = \cos\left(\frac{\pi\alpha}{2}\right) \tag{8}$$

The expression for $curl^\alpha$ acting on the function of one variable $\vec{F}(z) = F_x\vec{x} + F_y\vec{y} + F_z\vec{z}$, was presented in [1]:

$$\begin{aligned} curl^\alpha[\vec{F}(z)] = (BD_z^\alpha F_x(z) - AD_z^\alpha F_y(z))\vec{x} + \\ (AD_z^\alpha F_x(z) + BD_z^\alpha F_y(z))\vec{y} + \delta_{0\alpha} D_z^\alpha F_z(z)\vec{z}, \end{aligned} \tag{9}$$

where $\delta_{0\alpha}$ is the Kronecker delta and $D_z^\alpha \equiv_{-\infty} D_z^\alpha$ is the Riemann-Liouville fractional integral, defined [22] as

$$_{-\infty}D_z^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{z} (z-u)^{-\alpha-1} f(u)du, \; for\; \alpha < 0,\; z > a. \tag{10}$$

For the fractional derivatives with $\alpha > 0$, the above definition should be used with additional step $_{-\infty}D_z^\alpha f(z) = (d/dz)^n\; _{-\infty}D_z^{\alpha-m} f(z)$, where m is chosen so that $(\alpha - m)$ is negative.

To derive expressions (6) we used the presentation for $curl^\alpha$ of the function of two variables $\vec{F}(x,y) = \vec{z}e^{\imath ax+\imath by}$ [5, 6, 7],

$$\begin{aligned} curl^\alpha(\vec{z}e^{\imath ax+\imath by}) = e^{\imath ax+\imath by}\{\imath^\alpha A\frac{b}{(a^2+b^2)^{(1-\alpha)/2}}\vec{x} - \\ \imath^\alpha A\frac{a}{(a^2+b^2)^{(1-\alpha)/2}}\vec{y} + B(a^2+b^2)^{\alpha)/2}\vec{z}\} \end{aligned} \tag{11}$$

where A, B defined by equations (8). The choice of the appropriate branches of the terms $(a^2+b^2)^{(1-\alpha)/2}$, $(a^2+b^2)^{\alpha/2}$ depends on physical conditions.

A fractional wave is a wave propagating in the same direction as the original field. Application of $curl^\alpha$ (in the case of real values of α) reduces to the rotation of vectors $\vec{E}, \vec{H}$ by the angle $\pi\alpha/2$ in the plane perpendicular to the direction of wave propagation. It is interesting to note that $curl^\alpha$ yields the coupling of original electric and magnetic fields (6), (7). Thus the $curl^\alpha$ is an operator which changes the polarization of the field. In the case when the original field is linearly-polarized having some angle δ_0 with the axis x, and real values of α, the fractional field remains linearly polarized, but the polarization vector is rotated by the angle $\delta_\alpha = \delta_0 + \pi\alpha/2$. It can be shown, that for the complex values of α, application of $curl^\alpha$ to the linearly-polarized plane wave yields an elliptically polarized plane wave. Complex and higher order fractional curl operator in electromagnetics was discussed in [23].

3. REFLECTION PROBLEMS

Now we consider fractional solutions to reflection problems. Consider a classic 2D problem of the plane wave oblique incidence on a plane boundary located at $y = 0$. Assume that the incident wave is the sum of TM and TE waves defined by components $E_z = D_e e^{\imath k(x\cos\phi + y\sin\phi)}$ and $H_z = D_m e^{\imath k(x\cos\phi+y\sin\phi)}$ with amplitudes D_e, D_m. ϕ is the angle between axis x and the direction of wave propagation. The boundary is characterized by isotropic impedance BC,

$$\vec{n} \times \vec{E} = \eta\vec{n} \times (\vec{n} \times \vec{H}) \tag{12}$$

where $\vec{E} = \vec{E}^i + \vec{E}^r$, $\eta_0\vec{H} = \eta_0\vec{H}^i + \vec{H}^r$ $(y > 0)$ is the total field, which is the sum of the incident $(\vec{E}^i, \eta_0\vec{H}^i)$ and the reflected $(\vec{E}^r, \eta_0\vec{H}^r)$ waves and η is the surface impedance. The normal $\vec{n}$ of the impedance boundary coincides with the axis y. As special cases, the BC (12) generates a PEC boundary for $\eta = 0$, and a PMC boundary for $\eta = \infty$. Original solution can be described in terms of 2D reflection dyadic $\hat{R}$ [18]:

$$\vec{E}^r = \hat{R}\vec{E}^i \tag{13}$$

with $\hat{R}$ written in matrix form as

$$\hat{R} = \begin{pmatrix} -R_H & 0 \\ 0 & R_E \end{pmatrix} \tag{14}$$

Reflection coefficients R_E, R_H for impedance boundary are defined as

$$R_E = -\frac{1 - \eta/\eta_0 \sin\phi}{1 + \eta/\eta_0 \sin\phi}, \quad R_H = -\frac{1 - \eta_0/\eta \sin\phi}{1 + \eta_0/\eta\sin\phi} \tag{15}$$

We assume that the original solution $(\vec{E}, \vec{H})$, from which we will build the fractional one, is a known solution of this reflection problem with a certain value of impedance η.

The fractional field can be constructed in two ways:
(1) apply $curl^\alpha$ to the total field, i.e., to the sum of incident and reflected fields:

$$(\vec{E}^\alpha, \eta_0\vec{H}^\alpha) = (\imath k_0)^{-\alpha} curl^\alpha(\vec{E}^i + \vec{E}^r, \eta_0\vec{H}^i + \eta_0\vec{H}^r) \tag{16}$$

(2) apply $curl^\alpha$ only to the reflected field, while the incident field remains unchanged:

$$\begin{aligned} (\vec{E}^{\alpha,r}, \eta_0\vec{H}^{\alpha,r}) = (\imath k_0)^{-\alpha} curl^\alpha(\vec{E}^r, \eta_0\vec{H}^r) \\ (\vec{E}^{\alpha,i}, \eta_0\vec{H}^{\alpha,i}) \equiv (\vec{E}^i, \eta_0\vec{H}^i) \end{aligned} \tag{17}$$

Keeping in mind the properties of the operator $curl^\alpha$, it is interesting to find the types of solutions that the fractional field can represent in the case when the original solution is a solution of reflection problem. Having the representation for the fractional field (expressed via the original field components and FO α), the corresponding BC, which the fractional field satisfies, can be derived. We are interested in finding a model of the fractional boundary, which can adequately describe intermediate reflection properties of the fractional solution.

In the first case (16) the fractional field describes the original solution and the dual solution as special cases of $\alpha = 0$ and $\alpha = 1$, respectively.

Duality principle in the reflection problems states that the dual solution corresponds to
(i) PMC boundary when the original field is a solution for the PEC;
(ii) Impedance surface with impedance η^{-1} when the original solution is a solution for the impedance η;
(iii) Conductive surface when the original is a solution for the resistive surface [17].
So fractional solution will correspond to some fractional boundaries which are intermediate cases between the "original boundary" and the "dual boundary".

In the second case (17) fractional field also represents the solution of reflection problem. But in the limit case of $\alpha = 1$ we have the situation where the boundary is not "dual" — this boundary acts as some kind of a polarizer similar to the twist polarizer.

Application of $curl^\alpha$ to the incident field can be expressed in terms of dyadic $\widehat{L}^{\alpha,i}$:

$$(\imath k_0)^{-1} curl^\alpha \vec{E}^i = \widehat{L}^{\alpha,i} \vec{E}^i, \quad \text{for} \quad y \to +0 \tag{18}$$

where

$$\widehat{L}^{\alpha,i} = \begin{pmatrix} B & -A\sin\phi \\ A\sin^{-1}\phi & B \end{pmatrix} \tag{19}$$

For the reflected field, $(\imath k_0)^{-1} curl^\alpha \vec{E}^r = \widehat{L}^{\alpha,r} \vec{E}^r$, where

$$\widehat{L}^{\alpha,r} = e^{\imath\pi\alpha} \begin{pmatrix} B & -A\sin\phi \\ A\sin^{-1}\phi & B \end{pmatrix} \tag{20}$$

For the normal incidence ($\phi = \pi/2$) and real values of α, both dyadics $\widehat{L}^{\alpha,i}$, $\widehat{L}^{\alpha,r}$ represent rotation by the angle $\pi\alpha/2$.

Finally, reflection properties of the fractional boundary can be described by the reflection dyadic $\widehat{R}^\alpha$:

$$\vec{E}^{\alpha,r} = \widehat{R}^\alpha \vec{E}^{\alpha,i} \tag{21}$$

Now we analyze the two models (16), (17) of the fractional solution.

3.1. Model 1. Anisotropic impedance boundary

Here, the fractional field is

$$\vec{E}^{\alpha,i} = \widehat{L}^{\alpha,i}\vec{E}^i, \vec{E}^{\alpha,r} = \widehat{L}^{\alpha,r}\vec{E}^r \tag{22}$$

Substituting this into (21), and using the relation (13), we get the fractional reflection dyadic:

$$\widehat{R}^\alpha = \widehat{L}^{\alpha,r}\widehat{R}(\widehat{L}^{\alpha,i})^{-1} \tag{23}$$

Then $\widehat{R}^\alpha$ can be found as

$$\widehat{R}^\alpha = e^{\imath\pi\alpha} \begin{pmatrix} -B^2R_H + A^2R_E & -AB\sin\phi(R_H + R_E) \\ -AB\sin^{-1}\phi(R_H + R_E) & -A^2R_H + B^2R_E \end{pmatrix} \tag{24}$$

For $\alpha = 0$ and $\alpha = 1$ we obtain

$$\widehat{R}^\alpha|_{\alpha=0} = \begin{pmatrix} -R_H & 0 \\ 0 & R_E \end{pmatrix}, \ \widehat{R}^\alpha|_{\alpha=1} = \begin{pmatrix} -R_E & 0 \\ 0 & R_H \end{pmatrix} \tag{25}$$

that corresponds to the original solution and the dual solution with impedance η^{-1} (using the property that $R_H(\eta) = R_E(\eta^{-1})$). Formula (25) is in agreement with the duality principle.

For the normal incidence $\widehat{R}^\alpha$ can be written in simple form as $\widehat{R}^\alpha = R_E e^{\imath\pi\alpha}\widehat{I}$, where $\widehat{I}$ is the unit dyadic. If the original solution is a solution for PEC ($\eta = 0$), then

$$\widehat{R}^\alpha = -e^{\imath\pi\alpha}\widehat{I} \tag{26}$$

From this equation, impedance η_α can be obtained as

$$\eta_\alpha = \imath\eta_0 \tan\left(\frac{\pi\alpha}{2}\right) \tag{27}$$

It means that for the case when original boundary is PEC the fractional boundary can be modeled as an isotropic impedance boundary with the value of impedance expressed from α (27). The same formula was derived by Engheta in [1].

In general, fractional boundary for intermediate cases ($0 < \alpha < 1$) can be modeled by anisotropic impedance BC

$$\vec{n} \times \vec{E}^\alpha = \widehat{\eta}_\alpha \vec{n} \times (\vec{n} \times \vec{H}^\alpha) \tag{28}$$

where the impedance $\widehat{\eta}_\alpha$, in general, can be written as a tensor

$$\widehat{\eta}_\alpha = \begin{pmatrix} \eta_{11}^\alpha & \eta_{12}^\alpha \\ \eta_{21}^\alpha & \eta_{22}^\alpha \end{pmatrix} \tag{29}$$

For simplification purposes we consider important special case of the tensor (29) corresponding to $\eta_{12}^\alpha = \eta_{21}^\alpha = 0$. Then coefficients $\eta_{11}^\alpha, \eta_{22}^\alpha$ are expressed as

$$\eta_{11}^\alpha = \frac{E_z^\alpha}{H_x^\alpha},\ \eta_{22}^\alpha = \frac{E_x^\alpha}{H_z^\alpha} \tag{30}$$

Here, we assumed that TE and TM parts of the original field have the same amplitudes, i.e. $D_e = D_m$. Coefficients (30) can be obtained as

$$\eta_{11}^\alpha = -\frac{1}{\sin\phi}\eta_0\frac{e^{\imath\pi\alpha}(B-A)+BR_E+AR_H}{e^{\imath\pi\alpha}(B-A)-(BR_E+AR_H)},$$
$$\eta_{22}^\alpha = -\sin\phi\eta_0\frac{e^{\imath\pi\alpha}(B+A)+AR_E-BR_H}{e^{\imath\pi\alpha}(B+A)-(AR_E-BR_H)} \tag{31}$$

In the limit cases: we obtain a) $\alpha = 0$: $\eta_{11}^\alpha = \eta_{22}^\alpha = \eta$; b) $\alpha = 1$: $\eta_{11}^\alpha = \eta_{22}^\alpha = \eta^{-1}$.

If we assume that the original field is solution for the PEC boundary, i.e., $\eta = 0$, then we obtain

$$\eta_{11}^\alpha = \imath\frac{1}{\sin\phi}\eta_0\tan\left(\frac{\pi\alpha}{2}\right), \quad \eta_{22}^\alpha = \imath\sin\phi\eta_0\tan\left(\frac{\pi\alpha}{2}\right) \tag{32}$$

This equation for fractional anisotropic impedance generalizes (27) for the case of oblique incidence.

Numerical results for absolute values of fractional anisotropic impedance are shown in Figures 1–3. Figures 1–2 show absolute values of η_{11}^α, η_{22}^α as a function of α for fixed value of the original impedance $\eta/\eta_0 = 0.5$. For $\alpha = 1$ both elements of the fractional impedance equal to $\eta_{11}^\alpha/\eta_0|_{\alpha=1} = \eta_{22}^\alpha/\eta_0|_{\alpha=1} = 2$, that corresponds to isotropic impedance $(\eta/\eta_0)^{-1} = 2$. For intermediate values of $0 < \alpha < 1$ elements η_{11}^α, η_{22}^α are not equal to each other, while their absolute values are equal $|\eta_{11}^\alpha| = |\eta_{22}^\alpha|$ for the normal incidence only. For oblique incidence ($\phi \neq \pi/2$), the graphic of $|\eta_{11}^\alpha|$ has local maximas for $\alpha = 0.5$, while the graphic of $|\eta_{22}^\alpha|$ has no local maximas. As seen from (31) for the value $\alpha = 0.5$ ($A = B = 1/\sqrt{2}$) we obtain $\eta_{11}^\alpha/\eta_0 = 1/\sin\phi$ for any value of the original impedance. For the normal incidence the value of the original impedance $\eta = 1$ describes the case of no reflection (fully absorber) when $R_E = R_H = 0$; in this case the fractional impedance is an isotropic impedance with $\eta_{11}^\alpha/\eta_0 = \eta_{22}^\alpha/\eta_0 = -1$ for any value of α (see Fig. 3).

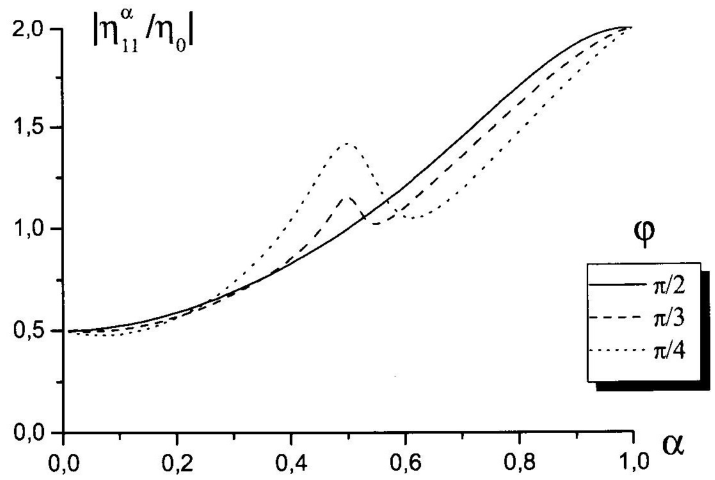

Figure 1. Absolute value of the fractional impedance $|\eta_{11}^\alpha/\eta_0|$, as a function of the FO α for the value of the original impedance $\eta/\eta_0 = 0.5$ and for different incidence angles ϕ.

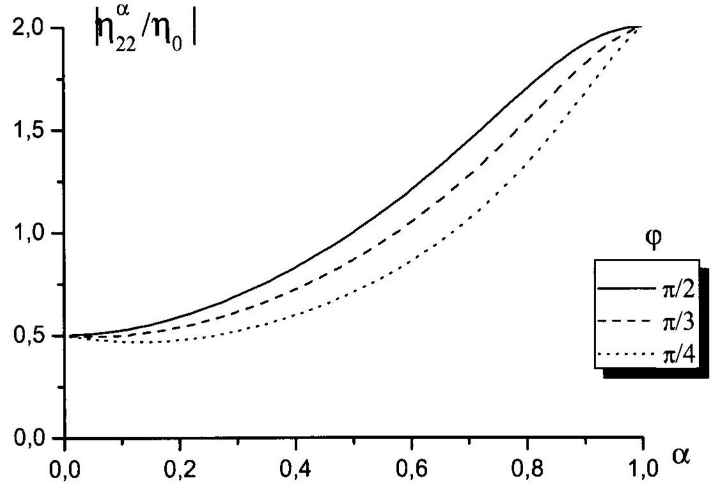

Figure 2. Absolute value of the fractional impedance $|\eta_{22}^\alpha/\eta_0|$ for the same values of parameters as on Fig. 1.

3.2. Model 2. Bi-isotropic Slab with PEC Backing

Here, the fractional solution is expressed as

$$\vec{E}^{\alpha,i} \equiv \vec{E}^i, \quad \vec{E}^{\alpha,r} = \vec{L}^{\alpha,r}\vec{E}^r \tag{33}$$

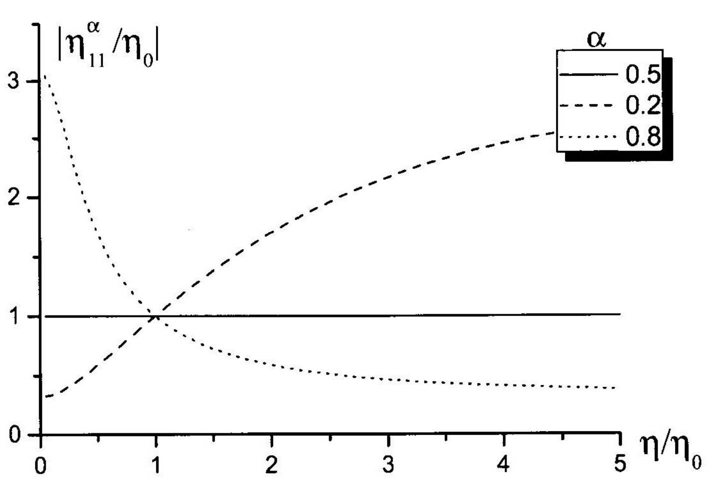

Figure 3. Absolute value of the fractional impedance $|\eta_{11}^{\alpha}/\eta_0|$, as a function of the original impedance η/η_0 for values of $\alpha = 0.2; 0.5; 0.8.$ for the normal incidence.

Then $\widehat{R}^{\alpha} = \widehat{L}^{\alpha,r}\widehat{R}$, and for the normal incidence we obtain

$$\widehat{R}^{\alpha} = \begin{pmatrix} -BR_H & AR_E \\ AR_H & BR_E \end{pmatrix} \tag{34}$$

In the case of PEC, we obtain

$$\widehat{R}^{\alpha} = -\begin{pmatrix} B & A \\ -A & B \end{pmatrix} \tag{35}$$

For the real α this is a rotation dyadic by the angle $(\pi - \pi\alpha/2)$.

Consider a problem of reflection from BI slab with PEC backing [21]. The upper half plane $(y > L)$ is an isotropic medium with parameters ϵ_0, μ_0, with the PEC plane located at $y = 0$. BI media $(0 < y < L)$ is defined by constitutive relations:

$$\vec{D} = \epsilon_2\vec{E} + \xi_2\vec{H},\ \vec{B} = \zeta_2\vec{E} + \mu_2\vec{H} \tag{36}$$

where parameters $\epsilon_2, \mu_2, \xi_2, \zeta_2$ are all scalars, ϵ_2 is the permittivity, μ_2 is the permeability. Parameters ξ_2, ζ_2 can be represented as

$$\xi_2 = (\chi_2 - \imath\kappa_2)\sqrt{\epsilon_0\mu_0},\ \zeta_2 = (\chi_2 + \imath\kappa_2)\sqrt{\epsilon_0\mu_0} \tag{37}$$

in terms of the Tellegen parameter χ_2 and the chirality κ_2. Reflection from this BI slab can be expressed in terms of reflection dyadic $\widehat{R}$:

$$\vec{E}^r = \widehat{R}\vec{E}^i \tag{38}$$

where for the case of normal incidence

$$\widehat{R} = \begin{pmatrix} R_{co} & R_{cr} \\ -R_{cr} & R_{co} \end{pmatrix} \tag{39}$$

Reflection coefficients R_{co}, R_{cr} are determined by parameters of the media η_2, χ_2 and thickness L [21]:

$$R_{co} = \frac{(\eta_0^2 - \eta_2^2)\sin^2 Q - \eta_0^2\cos^2\theta_2}{\eta_0^2\cos^2\theta_2 - (\eta_0^2 + \eta_2^2)\sin^2 Q + \imath\eta_0\eta_2\cos\theta_2\sin(2Q)}$$
$$R_{cr} = \frac{2\eta_0\eta_2\sin\theta_2\sin^2 Q}{\eta_0^2\cos^2\theta_2 - (\eta_0^2 + \eta_2^2)\sin^2 Q + \imath\eta_0\eta_2\cos\theta_2\sin(2Q)} \tag{40}$$

Here we denoted $Q = k_2 L\cos\theta_2$, $\eta_0 = \sqrt{\mu_0/\epsilon_0}$, $\eta_2 = \sqrt{\mu_2/\epsilon_2}$, $\xi_2 = \sin\theta_2$, and wavenumber $k_2 = \omega\sqrt{\epsilon_2\mu_2}$.

Comparing the reflection dyadic for the fractional solution (35) with coefficients for the BI slab (40), we conclude that the fractional boundary can be identified as BI slab if we choose the order α so that

$$R_{co} = -\cos\left(\frac{\pi\alpha}{2}\right),\ R_{cr} = \sin\left(\frac{\pi\alpha}{2}\right). \tag{41}$$

Such an α can be found from the following equation

$$\tan\left(\frac{\pi\alpha}{2}\right) = -\frac{R_{cr}}{R_{co}} = \frac{2\eta_0\eta_2\sin\theta_2\sin^2(k_2 L\cos\theta_2)}{(\eta_0^2 - \eta_2^2)\sin^2(k_2 L\cos\theta_2) - \eta_0^2\cos^2\theta_2} \tag{42}$$

For the given value of α, a fractional solution can be simulated with a BI slab backed by a PEC plane with parameters satisfying the above equation.

For the case when $\eta_2 = \eta_0$ we obtain from (42) the relation

$$\tan\left(\frac{\pi\alpha}{2}\right) = \frac{2\sin\theta_2\sin^2(k_2 L\cos\theta_2)}{\cos^2\theta_2} \tag{43}$$

In the special case of $\theta_2 = \pi/2$ $(\chi_2 = 1)$, $\alpha = \frac{2}{\pi}arctan(-2\eta_2(k_2 L)^2)$.

The results in Fig. 4, Fig. 5 and Fig. 6 show FO α as a function of the normalized thickness $k_2 L$ for various values of the Tellegen parameter χ_2 and normalized impedance η_2/η_0.

In the case when Tellegen parameter has the extreme value $\chi_2 = 1$, we obtain

$$\tan\left(\frac{\pi\alpha}{2}\right) = \frac{2\eta_0\eta_2(k_2 L)^2}{(\eta_0^2 - \eta_2^2)(k_2 L)^2 - \eta_0^2} \tag{44}$$

As seen from Fig. 4 α can be zero not only for the evident case when $k_2 L = 0$: $\alpha = 0$ also when $k_2 L\cos\theta_2 = \pi n$, $n = 0, 1, 2, \ldots$ It

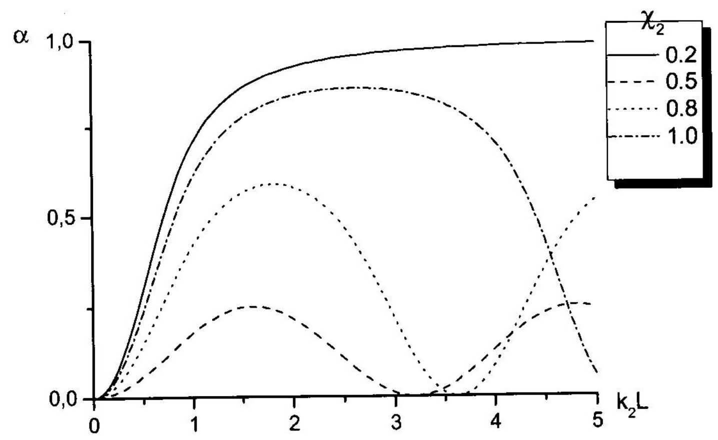

Figure 4. Fractional order α for normal incidence, as a function of the normalized thickness k_2L, for normalized impedance $\eta_2/\eta_0 = 1$ and different values of the Tellegen parameter χ_2 from 0.2 to 1.0.

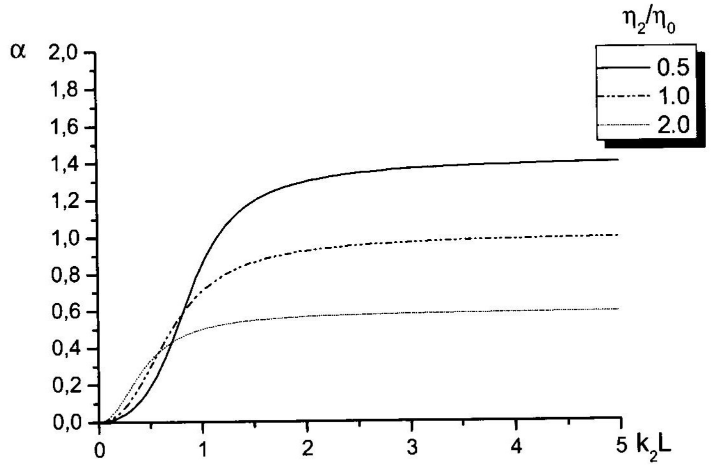

Figure 5. Fractional order α for normal incidence, as a function of the normalized thickness k_2L, for $\chi_2 = 1$ and different values η_2/η_0 from 0.5 to 2.0.

means, that for each value of the Tellegen parameter $\chi_2 < 1$ ($\theta_2 \neq \pi/2$) one can define the non-zero thicknesses L, so that $\alpha = 0$.

Values of α close to unity correspond to $R_{co} \to 0$, that is a twist polarizer effect.

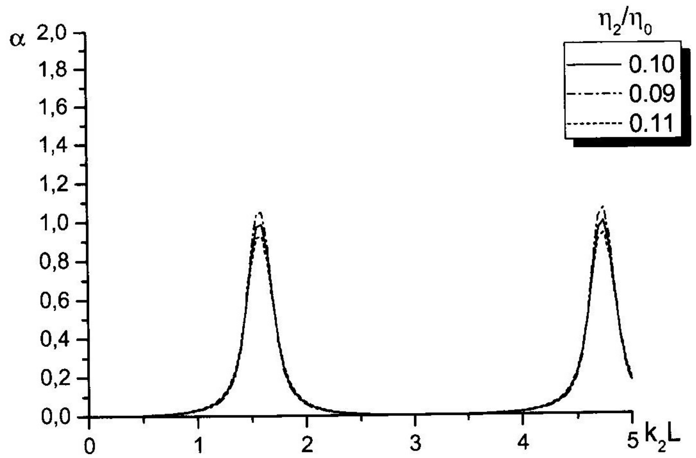

Figure 6. Fractional order α, as a function of the normalized thickness k_2L, for $\chi_2 = 0.1$ and various values of the η_2/η_0 near 0.1.

Considering transmission through BI slab, similar results can be observed, i.e., a BI slab acts as a polarization rotator for the transmitted wave [21]. In this case the same fractional model can be applied, with transmitted wave presented as a fractional field, but FO α will be related to chirality parameter κ_2, because the Tellegen parameter χ_2 has no effect on the transmitted wave.

4. CONCLUSION

In this paper, reflection properties of the fractional field are analyzed. Corresponding "'fractional"' or "'intermediate"' boundary is modeled by anisotropic impedance BC or BI slab with PEC backing. Dependence of FO on parameters of BI slab has been studied. It is shown that fractional boundary has the polarization transforming properties. Also a twist polarizer effect is described in terms of the limit value of FO $\alpha = 1$. Considered FOA in reflection problems is one of the possible applications of fractional operators in electromagnetic problems.

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