Fractional operators approach in reflection and diffraction problems

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Abstract – **Applications of fractional operators approach to electromagnetic waves reflection and diffraction problems are considered. Reflection properties of fractional or intermediate solutions obtained using fractional curl operator are analyzed. It is shown that this approach is a useful technique for the description of solutions to the reflection problems for some known mediums in terms of the fractional order. Also new features of boundaries corresponding to fractional solutions can be derived. In this paper intermediate boundaries are modelled with BI slab with a PEC backing. Finally, using fractional derivative new boundary conditions are introduced. The problem of diffraction on strip with fractional boundary conditions is solved.**

Keywords – **Fractional operators, diffraction, bi-isotropic medium.**

I. INTRODUCTION

During last fifteen years N. Engheta developed the tools of fractional operators to solve in various problems of electromagnetics [1], [2]. Fractional operators are defined as fractionalizations of some commonly used operators. In this paper fractional derivative (integral) or Riemann-Louville type and fractional curl operator $curl^{\alpha}$ with fractional order (FO) α [3] are considered.

We have applied the concept of the fractional operators to two dimensional (2D) problems of reflection from a media interface (the medium can be modeled by the impedance boundary conditions or can be a slab of some anisotropic or bi-anisotropic material, etc).

This approach is based on introducing new fractional field as the result of application of $curl^{\alpha}$ to the known "original" solution. Original solution takes into account the main features of solution of original reflection problem. Effect of the fractional order (FO) α yields a coupling of electric and magnetic fields that means changing the polarization of the original field. Similar effect is observed in bi-isotropic (BI) media [4]. Therefore the model of the fractional boundary using BI slab is considered. Fractional field can be treated as a solution of the problem with the same geometry but with new boundary. Properties of this fractional boundary will be defined by FO α and the original boundary.

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Fractional field was analyzed by many authors in various electromagnetic problems: propagation in chiral media [5], [6], waveguides [7], reflection problems with boundaries of impedance type [8-10]. Our interest to fractional operators in reflection problems is connected with the possibility of simple description of known boundaries as intermediate between canonical ones, and also obtaining the boundaries with new features.

N. Engheta introduced the concept of "Intermediate cases in electrodynamics" [1]. It means the following: if the function $f(y)$ and its first derivative $f^{(1)}(y)$ describe two canonical case of electromagnetic field, then fractional derivative ${}_{-\infty}D_y^{\alpha} f(y)$ defines an intermediate case between two canonical cases. In this paper fractional derivative, denoted as ${}_{-\infty}D_y^{\alpha} f(y)$, is defined via Riemann-Liouville integral [4], where fractional order ν between zero and unity ($0<\alpha<1$). Following this concept new fractional boundary conditions (FBC) are introduced, which are intermediate between boundary conditions (BC) of Dirichlet and Newmann type. FBC in respect to a function $U(x,y)$ on a plane boundary located at the plane $y=0$ are defined as

$$ {}_{-\infty}D_y^{\alpha} U(x,y)\,|_{y=0} = 0, \quad 0 \le \alpha \le 1 \qquad (1) $$

In this paper a problem of diffraction on finite strip with such FBC is solved.

II. FRACTIONAL CURL OPERATOR AND FRACTIONAL SOLUTIONS TO REFLECTION PROBLEMS

A. Properties of the fractional field

In this section we consider operator $curl^{\alpha}$, proposed in [3] as fractionalization of the conventional curl operator. The order α can be real, $0<\alpha<1$, however, complex values of α can be also considered. The $curl^{\alpha}$ was introduced to fractionalize the duality principle in electromagnetic theory [3]. A new electromagnetic field $(\vec{E}^{\alpha}, \eta_0 \vec{H}^{\alpha})$ is defined by applying $curl^{\alpha}$ to some known field $(\vec{E}^{0}, \eta_0 \vec{H}^{0})$, which is a solution of some electromagnetic problem with certain values of input parameters:

$$\vec{E}^{\alpha} \equiv (ik_0)^{-\alpha} curl^{\alpha} \vec{E}^0,$$
$$\eta_0 \vec{H}^{\alpha} \equiv (ik_0)^{-\alpha} curl^{\alpha} (\eta_0 \vec{H}^0) \tag{2}$$

Here $k_0 = \omega\sqrt{\varepsilon_0 \mu_0}$ is the wave number in the medium with permittivity ε_0 and permeability μ_0, $\eta_0 = \sqrt{\mu_0 / \varepsilon_0}$ is the intrinsic impedance of the medium. Time dependence is assumed to be $e^{-i\omega t}$. The field $(\vec{E}^0, \eta_0 \vec{H}^0)$ is referred to as an "original" solution. We name the field $(\vec{E}^{\alpha}, \eta_0 \vec{H}^{\alpha})$ as "fractional" or "intermediate" solution between the original and dual fields. If the original field is a given solution of Maxwell's equations, then the fractional field defined by expressions in Eqs. (2) represents another solution of Maxwell's equations with the same values of parameters ε_0, μ_0. For $\alpha = 0$ we obtain the original field $(\vec{E}^0, \eta_0 \vec{H}^0)$, and for $\alpha = 1$ the fractional field corresponds to the dual solution $(\eta_0 \vec{H}^0, -\vec{E}^0)$.

Consider a simple case when the original field is a uniform plane wave propagating along the direction given by the vector $\vec{l}(\cos\varphi, \sin\varphi, 0)$. This field can be obtained from two independent components, $E_z = D_e e^{ik(x\cos\varphi + y\sin\varphi)}$, $H_z = D_m e^{ik(x\cos\varphi + y\sin\varphi)}$, where D_e, D_m are the amplitudes. Other components are derived from these ones by using Maxwell's equations.

The components of the fractional field $\vec{E}^{\alpha}(E_x^{\alpha}, E_y^{\alpha}, E_z^{\alpha})$, $\vec{H}^{\alpha}(H_x^{\alpha}, H_y^{\alpha}, H_z^{\alpha})$ are expressed via the components of the original field [10] as

$$E_z^{\alpha} = \cos(\pi\alpha/2) E_z + \sin(\pi\alpha/2) \eta_0 H_z$$
$$\eta_0 H_z^{\alpha} = -\sin(\pi\alpha/2) E_z + \cos(\pi\alpha/2) \eta_0 H_z \tag{3}$$

where other components can be obtained from Maxwell's equations.

A fractional wave is a wave propagating in the same direction as the original field. Application of $curl^{\alpha}$ (in the case of real values of α) reduces to the rotation of field vectors $(\vec{E}, \eta_0 \vec{H})$ by the angle $\pi\alpha/2$ in the plane perpendicular to the direction of wave propagation. $curl^{\alpha}$ is an operator which changes the polarization of the field. In the case when the original field is linearly-polarized having some angle δ_0 with the axis x, and real values of α, the fractional field remains linearly polarized, but the polarization vector is rotated by the angle $\delta_{\alpha} = \delta_0 + \pi\alpha/2$. For the complex values of α, application of $curl^{\alpha}$ to the linearly-polarized plane wave yields an elliptically polarized plane wave.

B. Reflection problems

Consider a classic 2D problem of the plane wave oblique incidence on a plane boundary located at $y = 0$. Assume that the incident wave $(\vec{E}^i, \eta_0 \vec{H}^i)$ is the sum of TM and TE waves defined by components $E_z = D_e e^{ik(x\cos\varphi + y\sin\varphi)}$ and $H_z = D_m e^{ik(x\cos\varphi + y\sin\varphi)}$ with amplitudes D_e, D_m. φ is the angle between axis x and the direction of wave propagation. The boundary is characterized by isotropic impedance BC,

$$\vec{n} \times \vec{E} = \eta \vec{n} \times (\vec{n} \times \vec{H}) \tag{4}$$

where η is the surface impedance, the normal $\vec{n} = \vec{y}$.

Assuming that the original solution $(\vec{E}, \eta_0 \vec{H})$ is a known solution of this reflection problem with a certain value of impedance η, *we* build the fractional one as follows: apply $curl^{\alpha}$ to the reflected original field $(\vec{E}^r, \eta_0 \vec{H}^r)$ and remain the incident wave unchanged

$$(\vec{E}^{\alpha,r}, \eta_0 \vec{H}^{\alpha,r}) \equiv (ik_0)^{-\alpha} curl^{\alpha} (\vec{E}^r, \eta_0 \vec{H}^r)$$
$$(\vec{E}^{\alpha,i}, \eta_0 \vec{H}^{\alpha,i}) \equiv (\vec{E}^i, \eta_0 \vec{H}^i) \tag{5}$$

It can be shown that fractional field represents the solution of reflection problem. Our aim is to find corresponding BC, which the fractional field satisfies, and find a model of the fractional boundary, which can adequately describe intermediate reflection properties of the fractional solution.

The reflection properties of the fractional boundary can be described in terms of the reflection dyadic $\hat{R}^{\alpha}$:

$$\vec{E}^{\alpha,r} = \hat{R}^{\alpha} \vec{E}^{\alpha,i} \tag{6}$$

It can be shown that for normal incidence $\hat{R}^{\alpha}$ is expressed as

$$\hat{R}^{\alpha} = \begin{pmatrix} -BR_H & AR_E \\ AR_H & BR_E \end{pmatrix} \tag{7}$$

where $B = \cos(\pi\alpha/2)$, $A = \sin(\pi\alpha/2)$, and reflection coefficients R_E, R_H define reflection properties of the original impedance boundary

$$R_E = -\frac{1 - \eta/\eta_0 \sin\varphi}{1 + \eta/\eta_0 \sin\varphi}, \quad R_H = -\frac{1 - \eta_0/\eta \sin\varphi}{1 + \eta_0/\eta \sin\varphi} \tag{8}$$

Such reflection properties defined by Eq. (7) can be modelled by bi-isotropic (BI) slab ($0 < y < L$) with PEC backing [4]. BI medium is defined by constitutive relations:

$$\vec{D}=\varepsilon_2\vec{E}+\xi_2\vec{H}\,,\ \vec{B}=\zeta_2\vec{E}+\mu_2\vec{H} \qquad (9)$$

where $\xi_2=(\chi_2-i\kappa_2)\sqrt{\varepsilon_0\mu_0}$, $\zeta_2=(\chi_2+i\kappa_2)\sqrt{\varepsilon_0\mu_0}$ are presented in terms of the Tellegen parameter χ_2 and the chirality κ_2. Reflection from such a BI slab can be described by reflection dyadic $\hat{R}$

$$\hat{R}=\begin{pmatrix} R_{co} & R_{cr} \\ R_{cr} & R_{co} \end{pmatrix} \qquad (10)$$

where

$$\begin{aligned} R_{co}&=\frac{1}{Z}[(\eta_0^2-\eta_2^2)\sin^2 Q-\eta_0^2\cos^2\vartheta_2],\\ R_{cr}&=\frac{1}{Z}2\eta_0\eta_2\sin^2 Q\sin\vartheta_2, \\ Z&=\eta_0^2\cos^2\vartheta_2-(\eta_0^2+\eta_2^2)\sin^2 Q+i\eta_0\eta_2\cos\vartheta_2\sin 2Q \end{aligned} \qquad (11)$$

Here we denoted $\xi_2=\sin\vartheta_2$, $Q=k_2L\cos\vartheta_2$.

Comparing the reflection dyadic for the fractional solution with coefficients for the BI slab, we conclude that the fractional boundary can be identified as BI slab if we choose the order α so that $R_{co}=-\cos(\pi\alpha/2)$, $R_{cr}=\sin(\pi\alpha/2)$. Such an α can be found from the following equation

$$\tan(\frac{\pi\alpha}{2})=-\frac{R_{cr}}{R_{co}}=\frac{2\eta_0\eta_2\sin^2 Q\sin\vartheta_2}{(\eta_0^2-\eta_2^2)\sin^2 Q-\eta_0^2\cos^2\vartheta_2} \qquad (12)$$

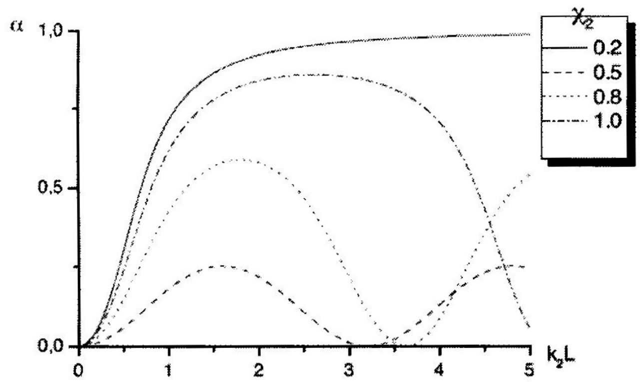

Fig. 1. Fractional order for normal incidence as a function of the normalized thickness k_2L, for normalized impedance $\eta_2/\eta_0=1$ and different values of the Tellegen parameter χ_2

For the case when $\eta_2=\eta_0$ we obtain from Eq. (11) the relation

$$\tan(\frac{\pi\alpha}{2})=\frac{2\sin^2(k_2L\cos\vartheta_2)\sin\vartheta_2}{\cos^2\vartheta_2} \qquad (13)$$

Numerical results for this case are given in Fig.1.

Values of α close to unity correspond to $R_{co}\to 0$, that is a twist polarizer effect.

III. Problem of Diffraction on Strip with Fractional Boundary Conditions

In this section new FBC defined by Eq. (1) are considered. These FBC are intermediate between BC of Dirichlet and Newmann type. Using these FBC we will build the solution of two-dimensional problem of diffraction of electromagnetic waves on a plane strip. Let the electromagnetic wave with a given intensity $U_0(x,y)$ be incident on a strip of width $2a$ ($-a<x<a$), located in the plane $y=0$, and infinite along the axis z. The field $U_0(x,y)$ is going from the half-space $y>0$ and is expressed as $U_0(x,y)=e^{ik(x\alpha_0+y\sqrt{1-\alpha_0^2})}$, where $\alpha_0=\cos\vartheta_0$, ϑ_0 - the incidence angle. For TM polarization case the function $U_0(x,y)$ denotes z-component of the electric field $U_0(x,y)\equiv E_z^0(x,y)$. The total field $U(x,y)$ is the sum of the incident $U_0(x,y)$ and the diffracted field $U_r(x,y)$. The solution $U_r(x,y)$ of the problem must satisfy Helmholtz equation everywhere out of the strip, radiation conditions, edge conditions, and also FBC on the strip $x\in(-a,a)$:

$$_{-\infty}D_{ky}^{\alpha}U(x,y)\,|_{y=0}=0\,,\, x\in(-a,a) \qquad (14)$$

The function $U_r(x,y)$ can be presented by using fractional Green's function G^α [3]:

$$U_r(x,y):=\int_{-a}^{a} f^{1-\alpha}(x')G^\alpha(x-x',y)dx' \qquad (15)$$

where G^α in 2D case is expressed as

$$G^\alpha(x,y)\equiv-\frac{i}{4}\ _{-\infty}D_{ky}^{\alpha}H_0^{(1)}(k\sqrt{x^2+y^2}) \qquad (16)$$

Here $H_0^{(1)}$ is the Hankel function of the first kind.

The function $f^{1-\alpha}(x)$, which describes the potential density in Eq. (15), is defined as the discontinuity of the fractional derivative of $U(x,y)$, i.e.

$$f^{1-\alpha}(x)={}_{-\infty}D_{ky}^{1-\alpha}U\,|_{y=+0}-{}_{-\infty}D_{ky}^{1-\alpha}U\,|_{y=-0} \qquad (17)$$

To obtain the function $f^{1-\alpha}(x)$ we use FBC in Eq. (14), and in terms of the Fourier transform of the function $f^{1-\alpha}(x)$ we obtain dual integral equations (DIE)

$$\begin{cases} \int_{-a}^{a} F^{1-\alpha}(\beta) e^{i\varepsilon\beta\xi}(1-\beta^2)^{\nu-1/2} d\beta = P(\xi), \ |\xi|<1 \\ \int_{-a}^{a} F^{1-\alpha}(\beta) e^{i\varepsilon\beta\xi} d\beta = 0, \ |\xi|>1 \end{cases} \quad (18)$$

where

$$P(\xi) = -4\pi i(1-\alpha_0^2)^{\alpha/2} e^{i\varepsilon\alpha_0\xi}$$

$$\tilde{f}^{1-\alpha}(\xi) \equiv a f^{1-\alpha}(a\xi), \ \xi = \frac{x}{a} \in [-1,1], \ \varepsilon = ka$$

$$F^{1-\alpha}(\beta) = \int_{-1}^{1} \tilde{f}^{1-\alpha}(\xi) e^{-i\varepsilon\beta\xi} d\xi \quad (19)$$

It is interesting to note, that for the special case $\alpha = 0.5$ from Eqs. (18) we can easily obtain the solution in the following form:

$$\tilde{f}^{0.5}(\xi) = -2\varepsilon(1-\alpha_0^2)^{1/4} e^{i\varepsilon\alpha_0\xi + i\pi/4} \quad (20)$$

and Fourier image

$$F^{0.5}(\beta) = -4(1-\alpha_0^2)^{1/4} e^{i\pi/4} \frac{\sin\varepsilon(\alpha-\alpha_0)}{\alpha-\alpha_0} \quad (21)$$

In general case the system in Eqs. (18) can be solved by using the method proposed in [11]. We present $\tilde{f}^{1-\alpha}(\xi)$ as series of Gegenbauer polynoms $C_n^{\alpha}(\xi)$ with the appropriate weight function:

$$\tilde{f}^{1-\alpha}(\xi) = (1-\xi^2)^{\alpha-1/2} \sum_{n=0}^{\infty} f_n^{\alpha} C_n^{\alpha}(\xi) \quad (22)$$

where f_n^{α} are unknown coefficients. Then $\tilde{f}^{1-\alpha}(\xi)$ satisfies edge condition of the following form

$$\tilde{f}^{1-\alpha}(\xi) = O((1-\xi^2)^{\alpha-1/2}), \ \xi \to \pm 1 \quad (23)$$

From Eq. (19) $F^{1-\alpha}(\beta)$ can be expressed as series

$$F^{1-\alpha}(\beta) = \frac{2\pi}{\Gamma(\alpha)} \sum_{n=0}^{\infty} (-i)^n f_n^{\alpha} \gamma_n^{\alpha} \frac{J_{n+\alpha}(\varepsilon\beta)}{(2\varepsilon\beta)^{\alpha}} \quad (24)$$

where $J_{n+\alpha}(x)$ are Bessel functions, $\gamma_n^{\alpha} = \dfrac{\Gamma(n+2\alpha)}{\Gamma(n+1)}$.

Substituting (22) into Eqs. (16), the second equation is satisfied automatically, and from the first equation we obtain the infinite system of linear algebraic equations (SLAE) to obtain the unknown coefficients f_n^{α}. By the method of reduction coefficients f_n^{α} can be found with any given accuracy [11]. With the known f_n^{α} density function $\tilde{f}^{1-\alpha}(\xi)$ and its Fourier image $F^{1-\alpha}(\beta)$ can be defined from Eqs. (22), (23), respectively.

IV. Conclusion

Proposed FOA can be a useful technique in the description of solutions to reflection problems for some known media in terms of FO α; besides, new boundaries with new features are obtained. Reflection properties of the fractional solution are analyzed. The model of BI slab with a PEC backing is proposed to describe fractional boundary. The twist-polarizer effect can be described in terms of specific value of FO α. New fractional BC are introduced for plane boundaries and the diffraction problem on a strip is solved.

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