FRACTIONAL BOUNDARY CONDITIONS IN SCATTERING PROBLEMS

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Abstract – **In this paper, we analyze some applications of the fractional boundary conditions (FBC) in the two-dimensional problems of wave reflection and diffraction. FBC are used to simulate reflection from dielectric slab where the fractional order depends on the layer parameters. The diffraction of an E-polarized electromagnetic field by a strip with FBC is studied. Numerical results are presented showing a comparison of the physical characteristics of the strip with FBC and impedance strip.**

I. INTRODUCTION

Tools of fractional calculus have found many applications in various problems of electromagnetics. Fractional operators defined as fractionalizations of some commonly used operators allow describing the intermediate states. Fractional paradigm was formulated by Engheta [1]. Following this idea, new fractional boundary conditions (FBC) were introduced in papers [2, 3]. In this paper FBC are defined by fractional derivatives of the tangential electric field components. For a boundary S located in the plane $y = 0$ in the E-polarization case FBC are

$$ {}_{-\infty}D_y^{\nu} E_z(x,y)\,|_{y\in S} = 0 \tag{1}$$

Here, the operator ${}_{-\infty}D_y^{\nu} f(y)$ is defined by the integral of Riemann-Liouville [4],

$$ {}_{-\infty}D_y^{\nu} f(y) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dy}\int_{-\infty}^{y}\frac{f(t)}{(y-t)^{\nu}}dt \tag{2}$$

The order of the fractional derivative (fractional order, FO) is assumed to be between 0 and 1. FBC describe an intermediate boundary between the perfect electric conductor (PEC) and the perfect magnetic conductor (PMC), obtained from FBC when FO equals to 0 and 1, respectively.

In this paper, FBC are applied in the modeling of the reflection from an infinite boundary and from a PEC-backed dielectric layer. In both cases the relations to define FO, ν, are derived. Finally, plane-wave diffraction by a fractional strip is studied. A fractional strip is introduced as strip with FBC involving fractional derivatives of the field components. Due to specific properties the fractional strip is compared with the well-known impedance strip. It is shown that for a wide range of input parameters the fractional strip has similar behavior as the impedance strip if the FO is chosen appropriately.

II. REFLECTION FROM BOUNDARIES DESCRIBED BY FRACTIONAL BOUNDARY CONDITIONS

Consider an incident E-polarized plane wave $\vec{E}^i(x,y) = \vec{z}E^i(x,y) = \vec{z}e^{-ik(x\alpha_0 + y\sqrt{1-\alpha_0^2})}$ coming from the upper space with ε_0, μ_0, where $\alpha_0 = \cos\theta_0$, θ_0 is the incidence angle and $k = 2\pi/\lambda$ is the wave number. The bottom space is defined by parameters ε_1, μ_1. Time dependence is assumed to be $e^{-i\omega t}$. It is known that reflection coefficient R in case of $N\to\infty$ [5]:

$$ R = -(1 - N^{-1}\frac{\varepsilon}{\varepsilon_0}\sin\varphi)/(1 - N^{-1}\frac{\varepsilon}{\varepsilon_0}\sin\varphi)\,,\ N = \sqrt{(\varepsilon\mu)/(\varepsilon_0\mu_0)} \tag{3}$$

Now we simulate the presence of the boundary by FBC where FO ν depends on ε_1, μ_1. FBC result in the reflection coefficient R_ν:

$$ R_\nu = -(-1)^{\nu} = -e^{i\pi\nu} \tag{4}$$

Comparing the coefficients (3) and (4) we find the equation to define FO ν

$$ \nu = \nu(\varepsilon,\mu) = \frac{1}{i\pi}\ln((1 + N^{-1}\frac{\varepsilon}{\varepsilon_0}\sin\theta_0)/(1 - N^{-1}\frac{\varepsilon}{\varepsilon_0}\sin\theta_0)) \tag{5}$$

It means that a half-space with parameters ε_1, μ_1 can be simulated by FBC with the FO ν defined from (5)

Similar way in case of normal incidence on a PEC-backed dielectric layer [5] of width d with parameters ε, μ can be replaced with the fractional boundary with FO ν defined from equation

$$\cot(kd) = \overline{\eta}\cot(\pi\nu/2),\ \overline{\eta} := \sqrt{\mu/\varepsilon}/\sqrt{\mu_0/\varepsilon_0},\ k = \omega\sqrt{\mu\varepsilon} \tag{6}$$

For the limit case $\nu = 1$ the fractional boundary corresponds to the layer where width satisfies the equation $kd = \pi/2$.

III. DIFFRACTION FROM A STIP

Consider a two-dimensional problem of electromagnetic wave diffraction by a strip located at the plane $y = 0$ and infinite along the axis z. The width of the strip is $2a$. E-polarization case is discussed. An incident plane wave is described by the function $\vec{E}^i(x,y) = \vec{z}E^i(x,y) = \vec{z}e^{-ik(x\alpha_0 + y\sqrt{1-\alpha_0^2})}$. Boundary conditions are FBC (1) with the surface $S = \{(x,y,z): y = 0, -a < x < a\}$. The function $E_z(x,y)$ denotes the z-component of the total electric field $E_z(x,y) = E_z^i + E_z^s$ — a sum of the incident plane wave $E_z(x,y)$ and the scattered field $E_z^s(x,y)$.

FBC yield to utilization of fractional Green's function (FGF) G^ν [6] and the fractional Green's theorem [6]. In this case the scattered field can be presented as [2]

$$E_z^s(x,y) := \int_{-a}^{a} f^{1-\nu}(x')G^\nu(x-x',y)dx' \tag{7}$$

where $f^{1-\nu}(x)$ is an unknown function which we name "fractional potential density". FGF G^ν is expressed in two-dimensional case as [2] as follows

$$G^\nu(x-x',y) = -i\frac{e^{\pm i\pi\nu/2}}{4\pi}\int_{-\infty}^{\infty} e^{ik[(x-x')\alpha + |y|\sqrt{1-\alpha^2}]}(1-\alpha^2)^{(\nu-1)/2}d\alpha\ , \tag{8}$$

Following the method presented in the works [2, 3] we present the scattered field $E_z^s(x,y)$ via the Fourier transform $F^{1-\nu}(\alpha) = a\int_{-1}^{1} f^{1-\nu}(a\xi)e^{-i\varepsilon\alpha\xi}d\xi$ of the fractional potential density $f^{1-\nu}(x)$:

$$E_z^s(x,y) = -i\frac{e^{\pm i\pi\nu/2}}{4\pi}\int_{-\infty}^{\infty} F^{1-\nu}(\alpha)e^{ik[x\alpha + |y|\sqrt{1-\alpha^2}]}(1-\alpha^2)^{(\nu-1)/2}d\alpha,\ \varepsilon = ka \tag{9}$$

Satisfying the function $E_z(x,y)$ FBC (1) we get IE [2]:

$$\frac{1}{\varepsilon}\int_{-\infty}^{\infty} F^{1-\nu}(\alpha)\frac{\sin\varepsilon(\alpha-\beta)}{(\alpha-\beta)}(1-\alpha^2)^{\nu-1/2}d\alpha = -4\pi e^{i\pi/2(1-\nu)}(1-\alpha_0^2)^{\nu/2}\frac{\sin\varepsilon(\beta+\alpha_0)}{\varepsilon(\beta+\alpha_0)} \tag{10}$$

In order to solve the IE we represent the density function $f^{1-\nu}(x)$ by a uniformly convergent series [2]

$$F^{1-\nu}(\alpha) = \frac{2\pi}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}(-i)^n f_n^\nu \beta_n^\nu \frac{J_{n+\nu}(\varepsilon\alpha)}{(2\varepsilon\alpha)^\nu} \tag{11}$$

where $J_{n+\nu}(\varepsilon\alpha)$ denotes Bessel function.. This presentation allows to satisfy the edge condition [7].
Substituting the series (11) into IE (10) after some transformation we can obtain [2] SLAE in respect to the coefficients f_n^ν. SLAE can be solved with the method of reduction, after that the fractional density $f^{1-\nu}(x)$ is evaluated and the electric field is obtained. Other physical characteristics such as radar cross section and surface current densities can be expressed as series in terms of the found coefficients f_n^ν.

An E-polarized plane wave incident on a fractional strip excites two surface currents – electric and magnetic. Similar current distributions are observed in the diffraction on an impedance strip: both boundaries support electric and magnetic surface currents which are perpendicular to each other. As a result of comparison of impedance boundary and fractional boundary in reflection problems discussed earlier we use the following relation between ν and η

$$\nu = \frac{1}{i\pi}\ln\frac{1-\eta\sin\vartheta_0}{1+\eta\sin\vartheta_0}, \qquad \eta = -i\frac{1}{\sin\vartheta_0}\tan(\frac{\pi\nu}{2}) \tag{12}$$

We consider the ratio $\zeta(x)$ for the fractional strip:

$$\zeta(x) := \frac{j_x^{\nu(m)}(x)}{j_z^{\nu(e)}(x)} = i\tan(\frac{\pi\nu}{2})\frac{A_\nu(x)}{B_\nu(x)}, \quad x \in (-a,a) \tag{13}$$

where electric and magnetic surface current densities are defined as $j_z^{\nu(e)}(x) = -(H_x(x,+0) - H_x(x,-0))$, $j_x^{\nu(m)}(x) = -(E_z(x,+0) - E_z(x,-0))$; functions $A_\nu(x)$, $B_\nu(x)$ can be obtained from fractional potential density function $f^{1-\nu}(x)$ [2, 3]. The ratio may depend on the coordinate x while the ratio $\eta = j_x^{\nu(m)}(x)/j_z^{\nu(e)}(x)$ for impedance strip is a constant η by the definition. However, for one special value $\nu = 0.5$ the IE can be solved analytically [2] and the function $\zeta(x)$ is a constant for any value of ε: $\zeta(x)|_{\nu=0.5} = -i\sin^{-1}\theta_0$. For the physical optics (PO) approximation ($\varepsilon \to \infty$) we can use asymptotic formulas for the integrals in equation (13) and the ratio $\zeta(x)$ is expressed analytically $\zeta(x) \sim -i\sin^{-1}\theta_0 \tan(\pi\nu/2) = \eta$, $\varepsilon \to \infty$. For finite boundaries in case of PO approximation we get exactly the same relation between the fractional order and the impedance(12). For arbitrary value of ε the ratio $\zeta(x)$ can be evaluated numerically. The less $\zeta(x)$ varies from constant (12) for $-a < x < a$ the more the fractional boundary has properties of an impedance boundary.

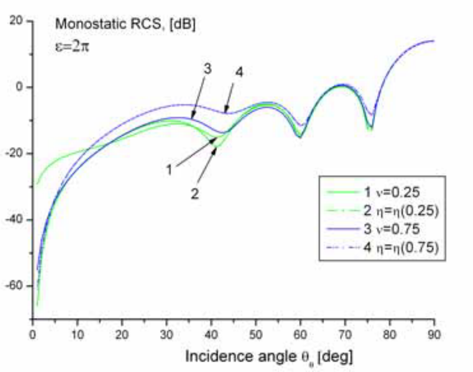

Figure 1. Monostatic RCS versus the incidence angle for $\varepsilon = 2\pi$. (1) fractional strip $\nu = 0.25$; (2) impedance strip for $\nu = 0.25$; (3) $\nu = 0.75$; (4) impedance corresponding to $\nu = 0.75$.

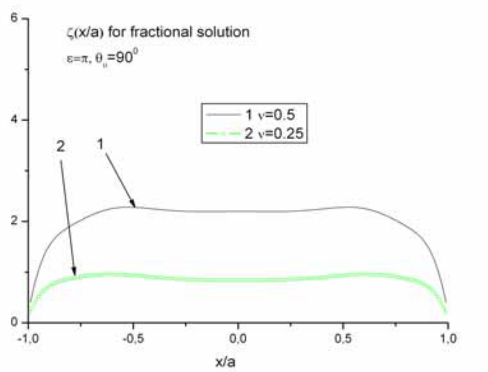

Figure 2. The ratio $\zeta(x)$ for the fractional strip for $\varepsilon = \pi$, $\theta_0 = 90^\circ$.

Figure 2 presents the graphic of the ratio $\zeta(x)$ for the fractional strip. For the wide range of the coordinate x the function $\zeta(x)$ is close to the value of impedance (12). It means that approximately a fractional strip can be treated as an impedance strip with pure imaginary impedance.

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