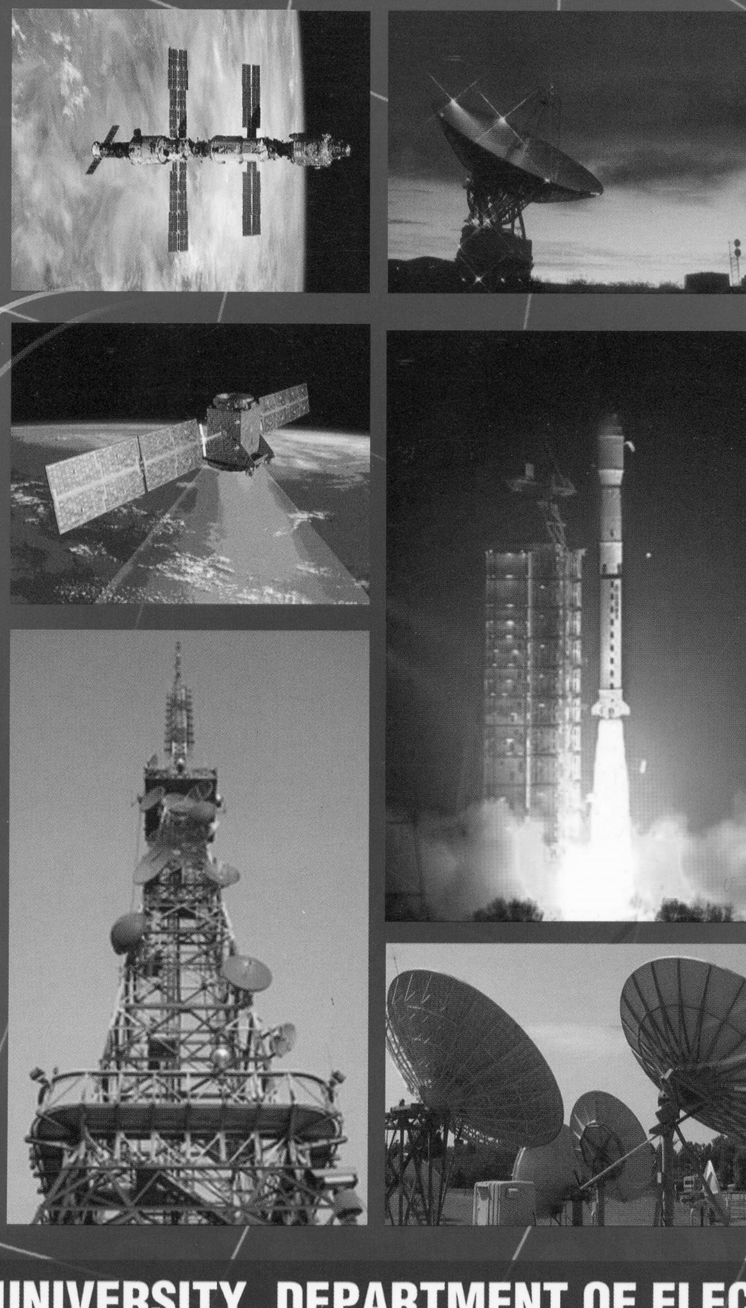
VTH INTERNATIONAL WORKSHOP ON ELECTROMAGNETIC WAVE SCATTERING



AKDENIZ UNIVERSITY, DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

ANTALYA, TURKEY

OCTOBER 22-25, 2008

Scattering Properties of the Strip with Fractional Boundary Conditions

M. V. Ivakhnychenko and E. I. Veliev
Institute of Radiophysics and Electronics NASU, ul. Proskury 12, Kharkov 61085, Ukraine

Abstract - New fractional boundary conditions (FBC) on plane boundaries are introduced. FBC involve fractional derivatives of the tangential electric field components. FBC describe intermediate boundary between perfect electric conductor (PEC) and perfect magnetic conductor (PMC). FBC are studied by the example of problem of diffraction of E-polarized electromagnetic field by a strip. The method of solving this problem is presented. It is shown that "fractional strip" has scattering properties similar to the well-known impedance strip. The relation between the fractional order and the value of impedance is derived.

1. INTRODUCTION

In this paper we analyze application of fractional boundary conditions (FBC) to diffraction by strip. Following the ideas of fractional paradigm in electrodynamics proposed by N. Engheta [1] we introduce FBC as intermediate case between well-known perfect electric conductor (PEC) and perfect magnetic conductor (PMC):

$$D^{\nu} U(r) = 0, \quad r \to S \tag{1}$$

where the fractional order (FO) ν is assumed to be between 0 and 1. D^{ν} in (1) denotes the fractional derivative of Riemann-Liouville type [2] and is applied along direction normal to the surface S. In diffraction problems the function $U(r)$ denotes tangential component of electric or magnetic field. PEC and PMC boundaries are obtained from FBC (1) when FO ν equal to 0 and 1, respectively.

Our interest is to study scattering properties of the two-dimensional strip with FBC on it. We refer a strip defined FBC as "fractional strip" in this paper. We will consider E-polarization case. The method of solving this problem will be presented later in this paper.

To understand properties of the boundaries described by FBC and relation to known boundaries we consider reflection problems first. It can be shown that infinite impedance boundary [3] with the value of impedance η can be simulated as boundary with FBC on it, i.e.

$${}_{-\infty}D^{\nu}_{ky} E_z = 0, y \to +0 \tag{2}$$

FBC result in the reflection coefficient $R_{\nu} = -(-1)^{\nu} = -e^{i\pi\nu}$. Comparing the coefficients for impedance and fractional boundaries we get the equation which relate FO ν and the impedance η:

$$\nu = \frac{1}{i\pi}\ln\frac{1-\eta\sin\vartheta_0}{1+\eta\sin\vartheta_0}, \qquad \eta = -i\frac{1}{\sin\vartheta_0}\tan(\frac{\pi\nu}{2}) \tag{3}$$

Keeping in mind the relation (3) for infinite boundaries we can expect that "fractional strip" of finite width has scattering properties similar to an impedance strip if the FO ν is related to impedance η as in (3). To compare "fractional" and impedance strips we independently solve both diffraction problems and compare physical characteristics such as cross sections and surface current densities.

2. PROBLEM FORMULATION

Consider a two-dimensional problem of electromagnetic wave diffraction by a strip of width $2a$ located at the plane $y=0$ and infinite along the axis z. An incident plane wave is described by the function $\vec{E}^i(x,y) = \vec{z}E^i(x,y) = \vec{z}e^{-ik(x\alpha_0 + y\sqrt{1-\alpha_0^2})}$, where $\alpha_0 = \cos\theta_0$, θ_0 is the incidence angle.

Boundary conditions are FBC (1) with the surface $S = \{(x,y,z): y=0, -a < x < a\}$:

$$_{-\infty}D_{ky}^{\nu} E_z(x,y)\,|_{y=0} = 0,\ x \in (-a,a) \tag{4}$$

Here for convenience the fractional derivative is applied with respect to the non-dimensional variable ky.

The function $E_z(x,y)$ denotes the z-component of the total electric field $E_z(x,y) = E_z^i + E_z^s$ — a sum of the incident plane wave $E_z(x,y)$ and the scattered field $E_z^s(x,y)$.

The solution $E_z(x,y)$ should satisfy the following conditions:

- $E_z(x,y)$ satisfies the Helmholtz equation everywhere outside the strip: $(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2)E_z(x,y) = 0$;
- $E_z^s(x,y)$ satisfies the radiation condition at infinity: $\lim_{r\to\infty}\sqrt{r}(\frac{\partial E_z^s}{\partial r} - iE_z^s) = 0$, $r = \sqrt{x^2+y^2}$;
- the Meixner's condition on the edges of the strip;
- $E_z(x,y)$ satisfies FBC on the strip surface (4).

3. SOLUTION

Utilizing the fractional Green's theorem [4] we present the scattered field via the fractional Greens' function G^ν [5, 6]

$$E_z^s(x,y) := \int_{-a}^{a} f^{1-\nu}(x')G^\nu(x-x',y)dx' \tag{5}$$

where $f^{1-\nu}(x)$ is an unknown function which we name "fractional potential density". FGF G^ν is expressed in two-dimensional case as $G^\nu(x-x',y) = -i\frac{e^{\pm i\pi\nu/2}}{4\pi}\int_{-\infty}^{\infty} e^{ik[(x-x')\alpha + |y|\sqrt{1-\alpha^2}]}(1-\alpha^2)^{(\nu-1)/2}d\alpha$.

Following the method presented in [5] we use the Fourier transform $F^{1-\nu}(\alpha) = a\int_{-1}^{1} f^{1-\nu}(a\xi)e^{-ika\alpha\xi}d\xi$ of the fractional potential density $f^{1-\nu}(x)$:

$$E_z^s(x,y) = -i\frac{e^{\pm i\pi\nu/2}}{4\pi}\int_{-\infty}^{\infty} F^{1-\nu}(\alpha)e^{ik[x\alpha+|y|\sqrt{1-\alpha^2}\,]}(1-\alpha^2)^{(\nu-1)/2}d\alpha,\ \varepsilon = ka \tag{6}$$

Satisfying the function $E_z(x,y)$ FBC (4) we get integral equation (IE) [5]:

$$\frac{1}{\varepsilon}\int_{-\infty}^{\infty} F^{1-\nu}(\alpha)\frac{\sin\varepsilon(\alpha-\beta)}{(\alpha-\beta)}(1-\alpha^2)^{\nu-1/2}d\alpha = -4\pi e^{i\pi/2(1-\nu)}(1-\alpha_0^2)^{\nu/2}\frac{\sin\varepsilon(\beta+\alpha_0)}{\varepsilon(\beta+\alpha_0)} \tag{7}$$

In order to solve the IE we represent the density function $f^{1-\nu}(x)$ by a uniformly convergent series [5] and the Fourier transform $F^{1-\nu}(\alpha)$ is expressed as

$$F^{1-\nu}(\alpha) = \frac{2\pi}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}(-i)^n f_n^\nu \beta_n^\nu \frac{J_{n+\nu}(\varepsilon\alpha)}{(2\varepsilon\alpha)^\nu} \tag{8}$$

where $J_{n+\nu}(\varepsilon\alpha)$ denotes Bessel function.

As a result of this presentation the edge conditions are satisfied in form

$$f^{1-\nu}(x) = O((1-x^2)^{\nu-1/2}),\ x\to\pm1 \tag{9}$$

Substituting the series (8) into IE (7) we obtain a system of linear algebraic equations (SLAE) in respect to the coefficients f_n^ν. SLAE is solved with the method of reduction, after that the fractional density $f^{1-\nu}(x)$ is evaluated and other physical characteristics can be obtained as series in terms of the found coefficients f_n^ν.

Analyzing IE (7) for fractional strip it is seen that for the special case of $\nu = 0.5$ the kernel becomes simple and the IE can be solved analytically for any value of $\varepsilon = ka$:

$$F^{0.5}(\alpha) = -4i(1-\alpha_0^2)^{1/4}e^{i\pi/4}\frac{\sin\varepsilon(\alpha+\alpha_0)}{\alpha+\alpha_0} \tag{10}$$

Another problem we consider in this paper is diffraction by impedance strip. Solution to this problem was given in [7]. In order to compare scattering properties of fractional and impedance strips we first consider surface currents existing in both cases. It can be shown that an E-polarized plane wave incident on a fractional strip excites two surface currents – electric and magnetic:

$$j_z^{\nu(e)} = -2i\cos(\frac{\pi}{2}\nu)\frac{\iota}{4\pi}\int_{-\infty}^{+\infty}F^{1-\nu}(\alpha)e^{ik\alpha x}(1-\alpha^2)^{\frac{\nu}{2}}d\alpha, \tag{11}$$

$$j_x^{\nu(m)} = -2\sin(\frac{\pi}{2}\nu)\frac{\iota}{4\pi}\int_{-\infty}^{+\infty}F^{1-\nu}(\alpha)e^{ik\alpha x}(1-\alpha^2)^{\frac{\nu}{2}-\frac{1}{2}}d\alpha$$

Similar current distributions are observed in the diffraction on an impedance strip: magnetic current is directed along the axis z and electric current is along the axis x

We introduce the ratio $\zeta(x)$ for the fractional strip:

$$\zeta(x) := \frac{j_x^{\nu(m)}(x)}{j_z^{\nu(e)}(x)},\quad x\in(-a,a) \tag{12}$$

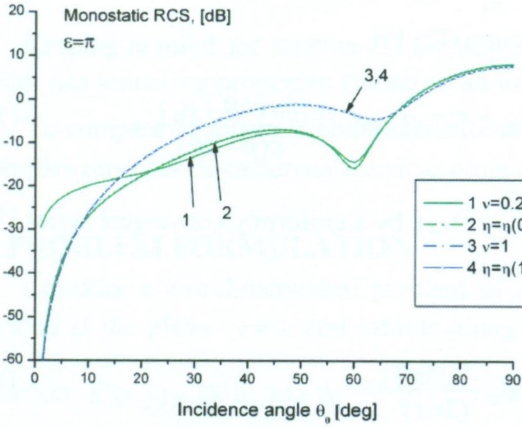


Figure 1. Monostatic RCS versus incidence angle for $\varepsilon = \pi$. (1) fractional strip with $\nu = 0.25$; (2) impedance strip with impedance η corresponding to $\nu = 0.25$; (3) fractional strip with $\nu = 0.75$; (2) impedance strip with impedance η corresponding to $\nu = 0.75$.

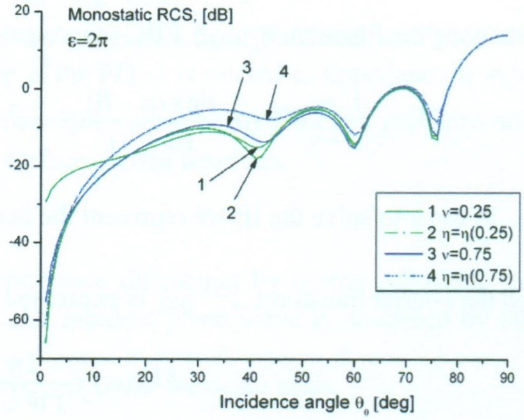


Figure 2. Monostatic RCS versus incidence angle for $\varepsilon = 2\pi$. (1) fractional strip with $\nu = 0.25$; (2) impedance strip with impedance η corresponding to $\nu = 0.25$; (3) fractional strip with $\nu = 1$; (2) impedance strip with impedance η corresponding to $\nu = 1$.

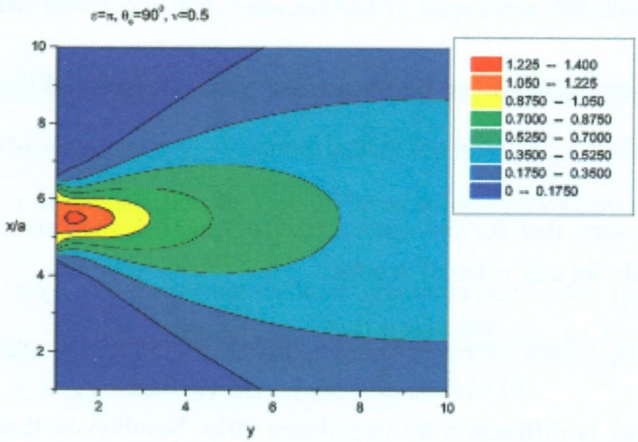


Figure 3. $|E_z|$ distribution for frequency parameter $\varepsilon = \pi$, incident angle $\theta_0 = 90^0$ and FO $\nu = 0.5$.

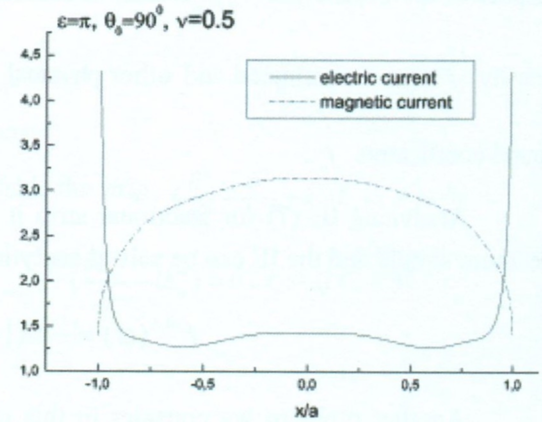


Figure 4. Magnetic and electric current densities for the same parameters as on Fig. 3.

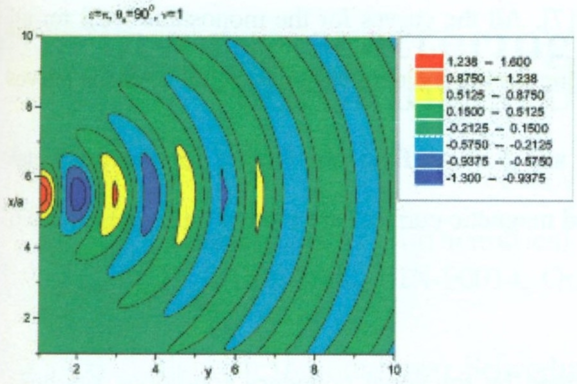


Figure 5. Distribution of real part of E_z for $\varepsilon = \pi$, $\theta_0 = 90^0$ and $\nu = 1$.

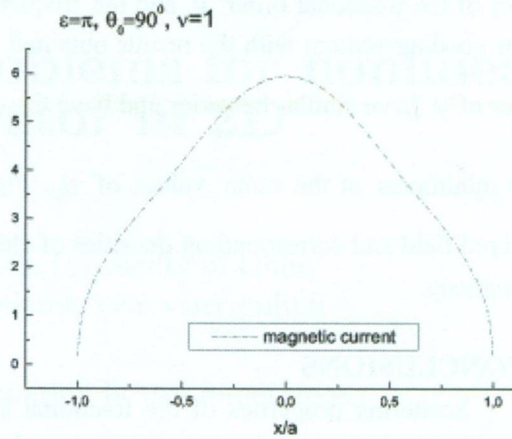


Figure 6. Magnetic current density for the same parameters as on Fig. 5. Electric current density equals to zero.

It should be noted that the ratio $\zeta(x)$ may depend on the coordinate x while the ratio $\eta = j_x^{\nu(m)}(x)/j_z^{\nu(e)}(x)$ for impedance strip is a constant η by the definition.

However, for one special value $\nu = 0.5$ the IE can be solved analytically [5] and the function $\zeta(x)$ is a constant for any value of $\varepsilon = ka$: $\zeta(x)|_{\nu=0.5} = -i\sin^{-1}\theta_0$ [6]. For the physical optics (PO) approximation ($\varepsilon = ka \to \infty$) we can use asymptotic formulas for the integrals in equation (12) and the ratio $\zeta(x)$ is expressed analytically $\zeta(x) \sim -i\sin^{-1}\theta_0 \tan(\pi\nu/2) = \eta$. For finite boundaries in case of PO approximation we get exactly the same relation between the fractional order and the impedance (3). For arbitrary value of ε the ratio $\zeta(x)$ can be eva luated numerically. The relation for the currents proves the fact that the fractional boundary conditions are similar to the impedance boundary conditions. The closer $\zeta(x)$ to the constant for $-a < x < a$ the more the fractional boundary has properties of an impedance boundary.

4. NUMERICAL RESULTS

In the far-zone ($kr \to \infty$) the scattered field is expressed as $E_z^r(x,y) \approx A(kr)?^{\,\nu}(\varphi)$, where

$$A(kr) = \sqrt{\frac{2}{\pi ? r}} e^{ikr - i\frac{\pi}{4}}, \quad ?^{\,\nu}(\varphi) = -\frac{i}{4}(\pm i)^{\nu} F^{1-\nu}(\cos\varphi)\sin^{\nu}\varphi \tag{13}$$

The upper sign is chosen for the values $\varphi \in [0, \pi]$, and the lower sign for $\varphi \in [\pi, 2\pi]$. The function $?^{\,\nu}(\varphi)$ denotes the radiation pattern (RP) of the scattered field and can be expressed via the coefficients f_n^{ν} found by solving SLAE.

Figures 1 and 2 show the comparison of the monostatic radar cross section (RCS) for different values of the fractional order ν and the frequency parameter $\varepsilon = ka$. The results for $\nu = 0$ and $\nu = 1$ are in good agreement with the results obtained earlier [7]. All the curves for the monostatic RCS for all values of ν have similar behavior and have the same value for the incident angle $\theta_0 = 90^0$. All the curves have minimums at the same values of θ_0. Figures 3 and 5 show field distribution for the fractional scattered field and corresponding densities of electric and magnetic currents are plotted on figures 4 and 6, respectively.

5. CONCLUSIONS

Scattering properties of the fractional strip defined by fractional boundary conditions has been analyzed. One important feature of the integral equations for the "fractional strip" is the fact that integral equation can be solved analytically for one special intermediate value of the FO equal to 0.5.

Detailed comparison analysis of the physical characteristics of the scattered fields for both fractional and impedance strips is presented. The fractional boundary supports both electric and magnetic currents. The analytical relation between the fractional order and the value of impedance is derived in cases of infinite boundaries and physical optics approximation. Similar to impedance boundary the ratio of surface current components is introduced for the strip with FBC. It is shown that in a wide range of input parameters the physical characteristics of the "fractional strip" are similar to the impedance strip.

REFERENCES

[1] Engheta, N. "Fractional Paradigm in Electromagnetic Theory," *a chapter in IEEE Press,* chapter 12, pp.523-553, 2000.

[2] Samko, S.G., A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach Science Publ., Langhorne, 1993.

[3] Senior, T.B.A. and J.L. Volakis, *Approximate Boundary Conditions in Electromagnetics*, IEE Press, London, 1995.

[4] Veliev, E.I. and N. Engheta, "Generalization of Green's Theorem with Fractional Differintegration," *2003 IEEE AP-S International Symposium & USNC/URSI National Radio Science Meeting*, 2003.

[5] Veliev, E.I., M.V. Ivakhnychenko, and T.M. Ahmedov, "Fractional boundary conditions in plane waves diffraction on a strip," *Progress In Electromagnetics Research, PIER 79*, 443–462, 2008.

[6] Ivakhnychenko, M. V., E. I. Veliev and T. M. Ahmedov, "Scattering Properties of the Strip with Fractional Boundary Conditions and Comparison with the Impedance Strip," *Progress In Electromagnetics Research C*, Vol. 2, pp. 189-205, 2008.

[7] Ikiz, T., S. Koshikawa, K. Kobayashi, E. I. Veliev, and A. H. Serbest, "Solution of the plane wave diffraction problem by an impedance strip using a numerical-analytical method: *E*-polarized case," *Journal of Electromagnetic Waves and Applications*, Vol. 15, No. 3, 315–340, 2001.