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ELEMENTARY FRACTIONAL DIPOLES

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Abstract – **In this paper the fractional curl operator $curl^{\alpha}$ is applied to obtain fractional sources as a generalization of the electric and magnetic Hertz dipoles. A1 physical meaning of the fractional curl operator in considered problem is shown: $curl^{\alpha}$ results in the coupling of the original electric and magnetic currents. For the values of the fractional order α between zero and one, the fractional sources can be treated as intermediate sources between the electric and magnetic sources. Expressions for the fractional fields and fractional sources are presented.**

I. INTRODUCTION

The fractional curl operator $curl^{\alpha}$ was introduced in [1]. Parameter α can be, in general, a complex number. We will use $curl^{\alpha}$ in three-dimensional radiation problems of excitation by elementary Hertz dipoles or Huygence elements. The fractional field $(\vec{E}^{\alpha},\vec{H}^{\alpha})$ is defined as a result of application of $curl^{\alpha}$ to the original field $(\vec{E},\vec{H})$ radiated from some current distribution:

$$\vec{E}^{\alpha}=(ik_0)^{-\alpha}curl^{\alpha}\vec{E};\quad \eta_0\vec{H}^{\alpha}=(ik_0)^{-\alpha}curl^{\alpha}(\eta_0\vec{H}) \tag{1}$$

where $k_0=\omega\sqrt{\varepsilon_0\mu_0}$ is the wavenumber, $\eta_0=\sqrt{\mu_0/\varepsilon_0}$ is the free space impedance, μ_0,ε_0 are the permittivity and permeability of the medium. Fractional currents are the currents corresponding to the fractional field and expressed through the original fields and the order α.

If $\alpha=0$ or $\alpha=1$ in the above expression, we get the original fields, $\vec{E}^{\alpha}|_{\alpha=0}=\vec{E}$, $\eta_0\vec{H}^{\alpha}|_{\alpha=0}=\eta_0\vec{H}$, or the dual fields, $\vec{E}^{\alpha}|_{\alpha=1}=\eta_0\vec{H}$, $\eta_0\vec{H}^{\alpha}|_{\alpha=1}=-\vec{E}$. Therefore, the fields (1) can be considered as "intermediate" or "fractional" solutions between the original fields and the dual fields. As will be shown later, fractional currents represent a coupling of original electric and magnetic currents. Fractional electric current can be treated as an intermediate source between the electric and magnetic currents.

II. FRACTIONAL SOURCES

Consider a field $(\vec{E}^{e},\vec{H}^{e})$ radiated from an elementary electric Hertz dipole [1] located at the origin and having the dipole moment $\vec{p}=p\vec{x}$, where $p=const$. This electric current is distributed with the density

$$\vec{j}_e(\vec{r})=\vec{p}\delta(r). \tag{2}$$

In the case when no magnetic current exists, the electromagnetic field can be expressed through the electric Hertz vector $\vec{A}^{e}$ [1]:

$$\vec{H}=rot\vec{A}^{e},\quad \vec{E}=-i\omega\mu\vec{A}^{e}+\frac{1}{i\omega\varepsilon}graddiv\vec{A}^{e}, \tag{3}$$

where $\vec{A}^{e}$ satisfies the Helmholtz equation:

$$\Delta\vec{A}^{e}+k^2\vec{A}^{e}=-\vec{j}_e \tag{4}$$

Electric Hertz vector, in its turn, can be expressed as

$$\vec{A}^e(\vec{r}) = \vec{p}G(\vec{r},0)\,, \tag{5}$$

where $G(\vec{r},\vec{r}\,') = \dfrac{-ie^{ik_0|r-r'|}}{4\pi|r-r'|}$ is the Green's function of the free space.

Fractional field $(\vec{E}^{\alpha,e},\vec{H}^{\alpha,e})$ is defined as $curl^\alpha$ applied to the field $(\vec{E}^e,\vec{H}^e)$ as shown in equation (1) and

$$\vec{H}^{\alpha,e} = rot\vec{A}^{\alpha,e}, \quad \vec{E}^{\alpha,e} = -\frac{\eta}{ik} rot\,\vec{H}^{\alpha,e}, \tag{6}$$

where

$$\vec{A}^{\alpha,e} = (ik_0)^{-\alpha} curl^\alpha \vec{A}^e = (ik_0)^{-\alpha} curl^\alpha[\vec{x}G(\vec{r},0)] = \frac{-i}{4\pi}(ik_0)^{-\alpha} curl^\alpha[\vec{x}\frac{e^{ik_0|r|}}{|r|}]. \tag{7}$$

The technique for obtaining a fractionalized operator was described in [1]. Following this recipe we can get the presentation for $curl^\alpha \vec{F}$ where the vector function $\vec{F}$ is expressed via exponents such as $\vec{F} = \vec{x}e^{iax+iby+icz}$:

$$\begin{aligned} curl^\alpha[\vec{x}e^{iax+iby+icz}] = \{&\vec{x}\frac{1}{k^2} i^\alpha(\delta_{0\alpha}a^2 + Bk^\alpha(b^2+c^2)) + \\ &\vec{y}\frac{1}{k^2} i^\alpha(\delta_{0\alpha}ab + k^\alpha B(Akc - Bab)) + \vec{z}\frac{1}{k^2} i^\alpha(\delta_{0\alpha}ac - k^\alpha B(Akb + Bac))\}e^{iax+iby+icz} \end{aligned}, \tag{8}$$

where $B = \cos(\pi\alpha/2)$, $A = \sin(\pi\alpha/2)$.

We can now derive the expressions for $\vec{A}^{\alpha,e}$ by using the expression (8) for $curl^\alpha$ and the representation of the Green's function in the form

$$G(x,y,z;0,0,0) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\xi_1 x - i\xi_3 z \pm\sqrt{\xi_1^2+\xi_3^2-k^2}\,y}}{\sqrt{\xi_1^2+\xi_3^2-k^2}} d\xi_1\xi_3\,, \tag{9}$$

where the upper sign is chosen for $y<0$ and the lower one for $y>0$.
The components of the vector $\vec{A}^{\alpha,e}$ are:

$$A_x^{\alpha,e} = [-B + (1 - k_0^{-\alpha}\delta_{0\alpha})\frac{x^2}{R^2}]\frac{e^{-ikR}}{4\pi R}, \; A_y^{\alpha,e} = [A\frac{z}{R} + (\pm k_0^{-\alpha}\delta_{0\alpha} \mp B)\frac{xy}{R^2}]\frac{e^{-ikR}}{4\pi R}, \tag{10}$$

$$A_z^{\alpha,e} = [\pm A\frac{y}{R} + (-k_0^{-\alpha}\delta_{0\alpha} + B)\frac{xz}{R^2}]\frac{e^{-ikR}}{4\pi R}.$$

In the spherical coordinate system (R,θ,φ) the fractional field $(\vec{E}^{\alpha,e},\vec{H}^{\alpha,e})$ $(0<\alpha<1)$ obtained from the field radiated by the electric dipole is expressed in the far-zone ($kR\to\infty$) as follows:

$$H_R^\alpha \cong 0,\; H_\theta^\alpha \cong \sin(\frac{\pi\alpha}{2})ik\sin\theta\frac{e^{-ikR}}{4\pi R},\; H_\varphi^\alpha \cong \cos(\frac{\pi\alpha}{2})ik\sin\theta\frac{e^{-ikR}}{4\pi R}, \tag{11}$$

$$E_R^\alpha \cong 0,\; E_\theta^\alpha \cong -ik\eta_0\cos(\frac{\pi\alpha}{2})\sin\theta\frac{e^{-ikR}}{4\pi R},\; E_\varphi^\alpha \cong ik\eta_0\sin(\frac{\pi\alpha}{2})\sin\theta\frac{e^{-ikR}}{4\pi R}.$$

For $\alpha = 0$, we have the original electric dipole:

$$H_R^e \cong 0,\ H_\theta^e \cong 0,\ H_\varphi^e \cong ik\sin\theta\frac{e^{-ikR}}{4\pi R};\qquad E_R^e \cong 0,\ E_\theta^e \cong -ik\eta_0\sin\theta\frac{e^{-ikR}}{4\pi R},\ E_\varphi^e \cong 0 \tag{12}$$

For $\alpha = 1$, we have the field $(\vec{E}^m, \vec{H}^m)$ radiated from the magnetic dipole:

$$H_R^m \cong 0,\ H_\theta^m \cong \frac{ik}{\eta_0}\sin\theta\frac{e^{-ikR}}{4\pi R},\ H_\varphi^m \cong 0;\qquad E_R^m \cong 0,\ E_\theta^m \cong 0,\ E_\varphi^m \cong ik\sin\theta\frac{e^{-ikR}}{4\pi R}, \tag{13}$$

where the magnetic dipole has the current density

$$\vec{j}_m(\vec{r}) = -\vec{j}_e(\vec{r}), \tag{14}$$

The fractional field $(\vec{E}^{\alpha,e}, \vec{H}^{\alpha,e})$ can be expressed as

$$H_\theta^\alpha \cong \cos(\pi\alpha/2)H_\theta^e + \sin(\pi\alpha/2)\eta_0 H_\theta^m,\qquad H_\varphi^\alpha \cong \cos(\pi\alpha/2)H_\varphi^e + \sin(\pi\alpha/2)\eta_0 H_\varphi^m. \tag{15}$$

The fractional field can be treated as a field generated by the fractional sources $(\vec{j}_{e,\alpha}, \vec{j}_{m,\alpha})$, which are a combination of the electric and magnetic dipoles:

$$\vec{j}_{e,\alpha} = B\vec{j}_e,\ \vec{j}_{m,\alpha} = -A\vec{j}_e \tag{16}$$

From the other hand, the fractional electric Hertz vector $\vec{A}^{\alpha,e}$ should satisfy the following Helmholtz equation:

$$\Delta\vec{A}^{\alpha,e} + k^2\vec{A}^{\alpha,e} = -\mathrm{curl}^\alpha(\vec{j}_e)$$

It means that the fractional field $(\vec{E}^{\alpha,e}, \vec{H}^{\alpha,e})$ is the field due to the electric current distribution given by

$$\vec{f}^{\alpha,e} = -\mathrm{curl}^\alpha(\vec{j}_e) \tag{17}$$

Fractional electric current $\vec{f}^{\alpha,e}$ (17) and fractional sources (16) are equivalent it the sense that these two source distributions radiate the same field.

III. CONCLUSION

We have analyzed a radiation from elementary fractional sources such as three-dimensional dipoles. The expressions for the "fractional" field have been determined with the fractional curl operator $curl^\alpha$ $(0 \le \alpha \le 1)$ applied to the fields radiated by an ideal electric dipole. In the limit cases of $\alpha = 0$ and $\alpha = 1$ the "fractional" field represents the fields due to conventional electric and magnetic dipoles, respectively. The expressions for the new "fractional" sources were derived from the "fractional" fields. We have shown that the fractional sources represent a coupling of electric and magnetic currents and can be treated as intermediate case between electric and magnetic sources.

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