APPLICATION OF FRACTIONAL BOUNDARY CONDITIONS IN DIFFRACTION BY PLANE SCREENS

M.V. Ivakhnychenko [1], **T.M. Ahmedov** [2], **E.I. Veliev** [1]

[1] Institute of Radiophysics and Electronics NASU, ul. Proskury 12, Kharkov 61085, Ukraine
[2] Institute of Mathematics NAS of Azerbaijan, 9 F.Agaeva, AZ 1141, Baku, Azerbaijan
e-mail: veliev@kharkov.ua

Abstract – **Problems of diffraction by plane screens described by fractional boundary conditions (FBC) are considered. FBC involves fractional derivative of tangential field components. FBC can be treated as intermediate case between well known boundary conditions for perfectly electric conductor (PEC) and perfectly magnetic conductor (PMC). A method to solve two-dimensional problems of scattering of E-polarized plane wave on a strip and half plane with FBC is proposed. The considered problems are reduced to coupled integral equations which are discretized using orthogonal polynomials. The method allows to obtain physical characteristics with any desired accuracy. One important feature of the considered integral equations is noted: these equations can be solved analytically for one special value of the fractional order equal to 0.5 for any frequency.**

I. INTRODUCTION

Fractional operators and transforms have been utilized in various electromagnetic problems by many authors. Fractional operators are defined as fractionalized ones from well-known operators. For example, fractional derivatives and integrals are generalizations of derivative and integral. Fractional curl operator defined in [1] and used by E.I. Veliev and Q.A. Navi in reflection problems [2, 3] is a fractionalized analogue of a curl operator used in many equations of mathematical physics.

Following the ideas of fractional paradigm in electrodynamics proposed by N. Engheta [4] fractional boundary conditions [5] are introduced to describe boundaries which can be treated as an intermediate case between well-known perfect electric conductor (PEC) and perfect magnetic conductor (PMC):

$$D_n^{\alpha} U(\vec{r})\,|_S = 0\,, \tag{1}$$

where the fractional order (FO) α is assumed to be between 0 and 1. D^{α} in (1) denotes the fractional derivative of Riemann-Luouville type on semi-infinite interval [6]. Fractional derivative D^{α} is applied along direction normal to the surface S. Usually, the function $U(r)$ denotes tangential component of electric or magnetic field.

From the first point of view, introduction of new boundary conditions must describe a new physical boundary world, from the other hand they must allow to build an effective computational algorithm to solve problems with the desired accuracy. For limit values $\alpha = 0$ and $\alpha = 1$ FBC (1) correspond to well known boundary conditions for PEC and PMC, respectively. FBC were analyzed in reflection problems by E.I. Veliev, N. Engheta [5, 7] where reflection coefficients have been presented. The reflection coefficient for this boundary has the module equal to 1 and the phase that depends on the value of the FO.

The purpose of this work is to build an effective analytic-numerical method to solve two-dimensional diffraction problems on boundaries described by FBC (1). The method is applied to model scattered objects – a strip and a half plane. As will be shown later this method allows to obtain a solution for one value $\alpha = 0.5$ in an explicit analytical form. For other values of $\alpha \in [0,1]$ the problems are reduced to solutions of infinite systems of linear algebraic equations (SLAE). Physical characteristics of the considered scattered objects can be found with any desired accuracy.

II. METHOD

Let us assume that an E-polarized plane wave, described by the function $\vec{E}^i = \vec{z}E_z^i(x,y) = \vec{z}e^{-ik(x\cos\theta + y\sin\theta)}$, is an incident field scattered by a strip or a half-plane located at the plane $y = 0$ and infinite along the axis z. θ is

978-1-4244-8860-5/10/$26.00 ©2010 IEEE

an incidence angle, $k = 2\pi/\lambda$ is a wave number. Here the time dependence is assumed to be $e^{-i\omega t}$ and deprecated throughout the paper. The total field $\vec{E} = \vec{z}E_z(x,y)$ must satisfy FBC (1)

$$D_{ky}^{\alpha} E_z(x,y) = 0\,,\ y \to \pm 0\,,\quad x \in L\,, \tag{2}$$

where $L = (-a,a)$ for a strip and $L = (0,\infty)$ for a half-plane. For convenience, fractional derivative is applied with respect to dimensionless variable ky.

The method is based on presentation of the scattered field $\vec{E}^s = \vec{z}E_z^s(x,y)$ with the help of fractional derivative of Green's function:

$$E_z^s(x,y) \equiv \int_L f^{1-\alpha}(x')G^{\alpha}(x-x',y)dx'\,. \tag{3}$$

This presentation is derived from fractional Green's theorem [5, 8]. In (3) the function $f^{1-\alpha}(x)$ is an unknown function named a fractional potential density, and $G^{\alpha}(x-x',y) = -i/4D_{ky}^{\alpha}H_0^{(1)}(k\sqrt{(x-x')^2+y^2})$ is a fractional derivative of Green's function.

For limit cases of the FO $\alpha = 0$ and $\alpha = 1$ presentation (3) corresponds to single-layer and double-layer potentials commonly used in diffraction problems.

Substituting $E_z(x,y)$ into FBC (2) we get the equation

$$\lim_{y\to 0} D_{ky}^{\alpha}\int_L f^{1-\alpha}(x')G^{\alpha}(x-x',y)dx' = -\lim_{y\to 0} D_{ky}^{\alpha}E_z^i(x,y)\,, \tag{4}$$

where the right part is a known function and $f^{1-\alpha}(x)$ is an unknown function.

It can be shown that the equation (4) can be induced to coupled integral equations (CIE)

$$\begin{cases} \int_{-\infty}^{\infty} F^{1-\alpha}(\beta)e^{ikd_L\xi\beta}(1-\beta^2)^{\alpha-1/2}d\beta = -4\pi e^{i\pi/2(1-\alpha)}\sin^{\alpha}\theta e^{-ikd_L\xi\cos\theta},\ \xi \in L, \\ \int_{-\infty}^{\infty} F^{1-\alpha}(\beta)e^{ikd_L\xi\beta}d\beta = 0, \qquad \xi \notin L. \end{cases} \tag{5}$$

where $F^{1-\alpha}(\beta)$ is a Fourier transform of the fractional potential density $f^{1-\alpha}(x)$, $d_L = a$ for $L = (-a,a)$, $d_L = 1$ for $L = (0,\infty)$.

For the limit cases of the fractional order $\alpha = 0$ and $\alpha = 1$ these equations are reduced to well known integral equations used for a PEC and PMC strips, respectively. In this paper the method to solve CIE (5) is proposed for any value of $\alpha \in [0,1]$.

CIE (8) can be solved analytically for one special case $\alpha = 0.5$. In this case we get the solutions for any value of k:

$$f^{0.5}(x) = -2ik\sin^{1/2}\theta e^{-ikx\cos\theta + i\pi/4} \text{ - for a strip,}$$

$$f^{0.5}(x) = -2\sin^{1/2}\theta e^{-ikx\cos\theta + i\pi/4} \text{ - for a half plane.}$$

In general case of α the solutions can be obtained numerically. The method is based on presentation of the unknown function $f^{1-\alpha}(x)$ via series of orthogonal polynomials with corresponded weight functions which allow to satisfy the edge conditions:

$$f^{1-\alpha}(x) = 1/a(1-(x/a)^2)^{\alpha-1/2}\sum_{n=0}^{\infty} f_n^{\alpha}\frac{1}{\alpha}C_n^{\alpha}(x/a)\,, \text{ for } L = (-a,a)\,, \tag{6}$$

$$f^{1-\alpha}(x) = e^{-x}x^{\alpha-1/2}\sum_{n=0}^{\infty} f_n^{\alpha}L_n^{\alpha-1/2}(2x)\,, \text{ for } L = (0,\infty)\,. \tag{7}$$

Gegenbauer polynomials are used for a strip, while for a half-plane Lager polynomials are used.

Substituting presentations (6), (7) into CIE (5) and using properties of discontinued integrals of Weber-Shafheitling (in case of a strip) and properties of Fourier presentations of Lager polynomials (for a half plane), CIE (5) can be reduced to an infinite SLAE in respect to unknown coefficients f_n^{α}. Finally, unknowns f_n^{α} can be obtained with any desired accuracy using the reduction method to solve SLAE. The H-polarization case for diffraction by a strip with FBC is considered in [9].

The method described above allows to obtain scattering characteristics of a strip or a half plane.

III. Numerical Results

In the far-zone ($kr \to \infty$) the scattered field is expressed as $E_z^s(r,\varphi) \approx A(kr)\Phi^{\alpha}(\varphi)$

$$A(kr) = \sqrt{\frac{2}{\pi kr}} e^{ikr - i\pi/4}, \ \Phi^{\alpha}(\varphi) = -\frac{i}{4} e^{\pm i\alpha\pi/2} F^{1-\alpha}(\cos\varphi)\sin^{\alpha}\varphi \qquad (8)$$

The upper sign is chosen for $\varphi \in [0,\pi]$, and the lower sign for $\varphi \in [\pi, 2\pi]$. The function $\Phi^{\alpha}(\varphi)$ defines the radiation pattern of the scattered field and can be expressed via the coefficients f_n^{α} found by solving SLAE.

Figures 1 and 2 show the comparison of the monostatic radar cross section (RCS) for a strip for different values of the fractional order α and the frequency parameter ka. The results for $\alpha = 0$ and $\alpha = 1$ are in good agreement with the results obtained earlier [10]. All the curves for the monostatic RCS for all values of α have similar behavior and have the same value for the incident angle $\theta = 90^0$.

More numerical results for diffraction by a strip with FBC one can find in papers [5, 11] where such physical characteristics as radiation pattern, RCS, surface current distributions and field distributions are shown.

An E-polarized plane wave incident on a strip with FBC excites two surface currents – electric and magnetic. Field distribution and surface current densities for a normal incidence are shown on figure 3 and 4, respectively. Similar current distributions are observed in the diffraction on an impedance strip. Comparison of the scattered properties of a strip with FBC and an impedance strip is presented in [11].

IV. Conclusion

The method to solve differintegral equation of a special kind, to which problems of diffraction by boundaries described by FBC, is proposed. The method is considered on two boundaries – a strip and a half plane with FBC when the fractional order varies from 0 to 1. FBC include boundary conditions for PEC and PMC as special cases of $\alpha = 0$ and $\alpha = 1$ so that the obtained solutions extend well known solutions for ideal strips and half planes. The proposed method is based on application of orthogonal polynomials. Gegenbauer polynomials orthogonal on interval $(-1,1)$ are utilized for a strip, while Lager polynomials orthogonal on interval $(0,\infty)$ are used for a half plane.

One important feature of the considered integral equations is noted: these equations can be solved analytically for one special intermediate value of the FO $\alpha = 0.5$ and it can be done for any value of the frequency.

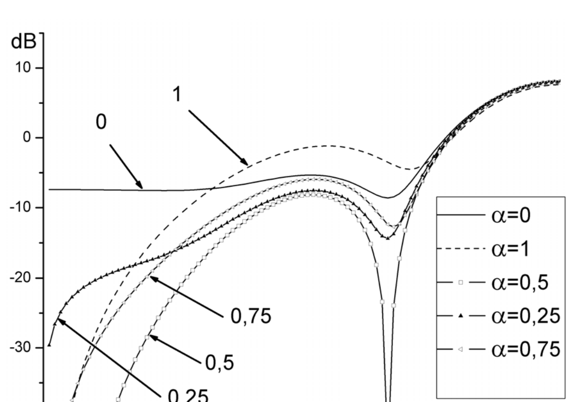

Fig. 1. Monostatic radar cross section as a function of incidence angle for $ka = \pi$ and different values of α.

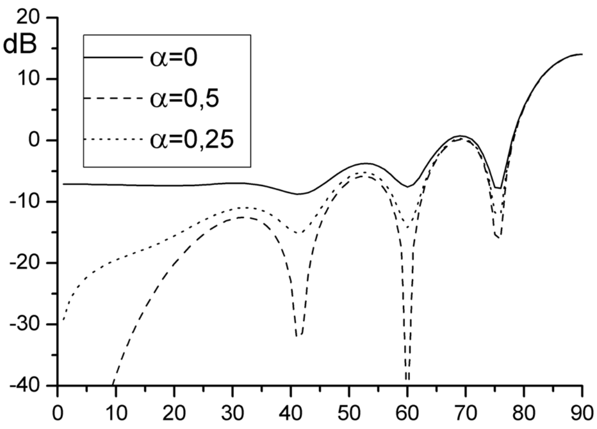

Fig. 2. Monostatic radar cross section as a function of incidence angle for $ka = 2\pi$ and different values of α.

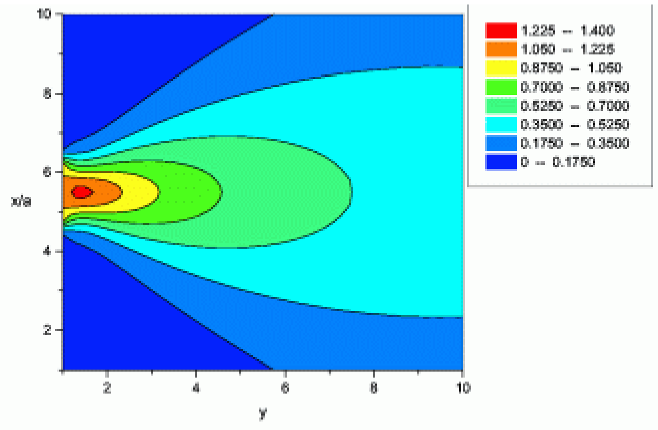

Fig. 3. $|E_z|$ distribution for frequency parameter $ka = \pi$, incident angle $\theta = 90^0$ and $\alpha = 0.5$.

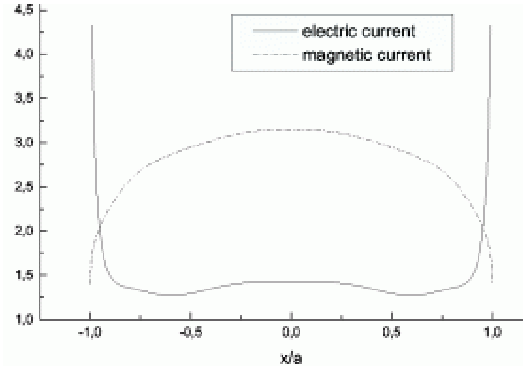

Fig. 4. Magnetic and electric current densities for the same parameters as on Fig. 3.

REFERENCES

[1.] N. Engheta, "Fractional curl operator in electromagnetics," *Microwave and Optical Technology Letters*, vol. 17, no. 2, pp. 86-91, 1998.

[2.] M. V. Ivakhnychenko, E. I. Veliev, T. M. Ahmedov, "Fractional operators approach in electromagnetic wave reflection problems," *Journal of electromagnetic waves and applications*, vol. 21, no. 13, pp. 1787-1802, 2007.

[3.] Q. A. Naqvi, A. A. Rizvi, "Fractional dual solutions and corresponding sources," *Progress In Electromagnetics Research*, vol. 25, pp. 223-238, 2000.

[4.] N. Engheta, "Fractional Paradigm in Electromagnetic Theory," *a chapter in IEEE Press,* chapter 12, pp.523-553, 2000.

[5.] E. I. Veliev, M. V. Ivakhnychenko, T. M. Ahmedov, "Fractional boundary conditions in plane waves diffraction on a strip," *Progress In Electromagnetics Research*, vol. 79, pp. 443-462, 2008.

[6.] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach Science Publ., Langhorne, 1993.

[7.] N. Engheta, "Fractionalization methods and their applications to radiation and scattering problems," Mathematical methods in electromagnetic theory, Kharkiv, 2000, pp. 34-40.

[8.] E. I. Veliev and N. Engheta, "Generalization of Green's Theorem with Fractional Differintegration," *2003 IEEE AP-S International Symposium & USNC/URSI National Radio Science Meeting*, 2003.

[9.] T. M. Ahmedov, "The solution of H-polarized plane wave diffraction by a plane strip based on fractional boundary conditions," *Proceedings of NAN of Azerbaijan*, vol. L X11, no. 5-6, pp. 8-18, 2006.

[10.] T. B. A. Senior, J. L. Volakis, *Approximate Boundary Conditions in Electromagnetics*, IEE Press, London, 1995.

[11.] M. V. Ivakhnychenko, E. I. Veliev, T. M. Ahmedov, "Scattering Properties of the Strip with Fractional Boundary Conditions and Comparison with the Impedance Strip," *Progress In Electromagnetics Research C*, vol. 2, pp. 189-205, 2008.