Theory of two-dimensional wave diffraction by a polygonal cylinder

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In this note we propose a rigorous, effective numerical method for the solution of the problem of wave diffraction by an ideally conducting polygonal cylinder. This method can be summarized by saying that the determination of the scattered field by each face of the polygonal cylinder (the total scattered field is the superposition of these fields) reduces to the solution of a system of coupled conjugate integral equations with respect to the Fourier transformations of functions describing the surface current densities on the faces. The method of moments in combination with the semi-inversion method is used to solve these equations. As a result, the problems reduces to a coupled system of linear algebraic equations, which are solved by the reduction method. The unknown terms in these equations are the coefficients in terms of which the surface current density on the faces is represented in the complete system of Gegenbauer polynomials with a weight factor which takes into account the behavior of the current at the vertices of the polygon (at the edges).

The individual questions examined in this paper have been discussed in a number of other studies (in particular, in Refs. 1–6).

1. Let a plane H-polarized electromagnetic wave[1)] $H_z^0 = e^{ik(\alpha_0 x + \sqrt{1-\alpha_0^2}\, y)}$ ($k = 2\pi/\lambda$, where λ is the wavelength, and $\alpha_0 = \cos\vartheta_0$), which varies in time as $e^{-i\omega t}$, be incident upon a polygonal cylinder at an angle ϑ_0. We assume that the cylinder is unbounded in the Oz direction and has a cross section in the shape of a regular polygon with equal angles at the vertices.

Let us introduce the following notion (see Fig. 1): N is the number of faces of the cylinder, $X_nO_nY_n$ are the lo local coordinate systems which are associated with the faces (n = 1, 2, ..., N), the angles $(\varphi_j)_{j=1}^N$ specify the orientation of the faces with respect to the O_nY_n axes, $(2a_j)_{j=1}^N$ are their widths, and $(\vartheta_j)_{j=1}^N$ are the angles of incidence in the $X_nO_nY_n$ coordinate systems.

The problem under consideration involves the determination of the H_z component of the field scattered by the polygonal cylinder, which must satisfy the wave equation outside the surface of the cylinder, the radiation condition at infinity, the condition that the energy in any bounded volume of space be finite (the Meichsner condition), and the Neumann boundary conditions at the surface of the cylinder.

The total field is represented in the form $H_z = H_z^0 + \sum_{s=1}^{N} H_z^s$, where H_z^s describes the scattered field produced by the surface current on the s-th face. In the local coordinate systems these fields will be sought in the form[6]

$$H_z^s = \frac{|y_s'|}{y_s'} \int_{-\infty}^{\infty} h_s(\alpha)\, e^{ik(\alpha x_s' + \sqrt{1-\alpha^2}\,|y_s'|)}\, d\alpha. \tag{1}$$

The unknown terms $\{h_s(\alpha)\}_{s=1}^N$ are the Fourier transformation of the functions $\{\mu_s(x_s')\}_{s=1}^N$ which describe the surface current densities on the faces:

$$\mu_s(x_s') = \int_{-\infty}^{\infty} h_s(\alpha)\, e^{ik\alpha x_s'}\, d\alpha, \quad x_s' \in (-a_s, a_s).$$

In (1) we have chosen that branch of the function $\sqrt{1-\alpha^2}$ for which in the limit $|\alpha| \to \infty$ $\operatorname{Im}\sqrt{1-\alpha^2} > 0$ along the real axis.

2. To determine the unknown terms $\{h_s(\alpha)\}_{s=1}^N$, we subject the total field to the Neumann boundary condition on the surface of the cylinder

$$\frac{\partial}{\partial n}\Big(H_z^0 + \sum_{s=1}^{N} H_z^s\Big)\Big|_L = 0, \quad L = \bigcup_{s=1}^{N} L_s, \tag{2}$$

where L_s is the contour of the face in the $X_sO_sY_s$ plane, and the normal n is oriented along the positive O_sY_s axis.

Equation (2) can be satisifed if the field H_z^s is written in the isolated coordinate system which corresponds to the face with the number j (see Fig. 1). We then obtained from (2) a coupled system of conjugate integral equations for the unknown terms $\{h_j(\alpha)\}_{j=1}^N$

$$\int_{-\infty}^{\infty} h_j(\alpha)\sqrt{1-\alpha^2}\, e^{i\varepsilon_j \alpha\eta_j}\, d\alpha = -\frac{2\pi}{k} A_j(\alpha_0)\, e^{i\varepsilon_j \eta_j \beta_j(\alpha)}$$

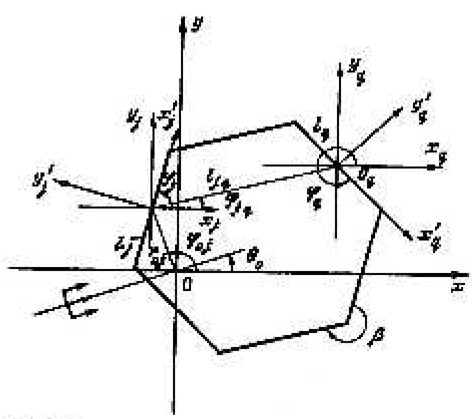

FIG. 1

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$$-\sum_{q=1,q\neq j}^{N}\int_{-\infty}^{\infty} h_q(\alpha)A_{jq}(\alpha)e^{i\epsilon_j\alpha\eta_j B_{jq}(\alpha)+ikl_{jq}D_{jq}(\alpha)}d\alpha, \quad |\eta_j|<1,$$

$$\int_{-\infty}^{\infty} h_j(\alpha)e^{i\epsilon_j\alpha\eta_j}d\alpha=0, \quad |\eta_j|>1. \tag{3}$$

where $\epsilon_j = ka_j$, $\eta_j = x_j/a_j$ is the dimensionless coordinate, L_{jq} is the distance between the centers of the faces with the numbers j and q, and

$$A_{jq} = -\alpha\sin(\varphi_j-\varphi_q) - \sqrt{1-\alpha^2}\cos(\varphi_j-\varphi_q);$$

$$A_j(\alpha_0) = \alpha_0\cos\varphi_j - \sqrt{1-\alpha_0^2}\sin\varphi_j,$$

$$B_{jq} = \alpha\cos(\varphi_j-\varphi_q) - \sqrt{1-\alpha^2}\sin(\varphi_j-\varphi_q);$$

$$B_j(\alpha_0) = -\alpha_0\sin\varphi_j - \sqrt{1-\alpha_0^2}\cos\varphi_j,$$

$$D_{jq}(\alpha) = -\alpha\cos(\varphi_q-\varphi_{jq}) + \sqrt{1-\alpha^2}\sin(\varphi_q-\varphi_{jq}).$$

The homogeneous equations in the system (3) are obtained by continuation of the functions $\mu_j(x'_j)$ by zero outside the interval $(-a_j, a_j)$.

The functions $\{h_j(\alpha)\}_{j=1}^N$ must satisfy, in addition to the system of equations (3), the relations $\int_{-\infty}^{\infty}\alpha|h_j(\alpha)|^2 d\alpha < \infty$, $j = 1, 2, \ldots, N$, which follow from the condition that the energy of the scattered waves be finite in any bounded region of space. It can be shown[7] that the solution of the system of equations (3) exists and is unique in the class of functions that satisfy this condition. The latter condition also guarantees the validity of the transformations carried out below.

3. To solve the system of equations (3), we represent the functions $\{\mu_j(\eta_j)\}_{j=1}^N$ in the form of uniformly converging series in the complete orthogonal system of Gegenbauer polynomials[8] $\{C_m^{\gamma+1/2}(\eta_j)\}_{m=0}^{\infty}$ with a weight factor which takes into account the behavior of the functions $\{\mu_j(\eta_j)\}_{j=1}^N$ at the ends of the interval $\eta_j \in [-1,1]$ (on the edge):

$$\mu_j(\eta_j) = (1-\eta_j^2)^{\nu}\sum_{m=0}^{\infty}\mu_m^j C_m^{\nu+1/2}(\eta_j) \tag{4}$$

Here $\{\mu_m^j\}_{m=0}^{\infty}$ are the new unknown coefficients, and $\nu = \pi/\beta$, where the angle β specifies the angle at the vertices of the polygon (see Fig. 1).

Representing the Fourier transformations of the functions $\{\mu_j(\eta_j)\}_{j=1}^N$ in the form of the sums of even and odd terms $h_j(\pm\alpha) = {}^1/_2[h_j^+(\alpha) \pm h_j^-(\alpha)]$ and using (4), we obtain representations for the functions $\{h_j^{\pm}(\alpha)\}_{j=1}^N$ in the form of uniformly converging series in the Bessel functions

$$h_j^+(\alpha) = \frac{4a_j\pi}{\Gamma(\nu+1/2)}\sum_{m=0}^{\infty}\mu_{2m}^j\frac{(-1)^m}{(2m)!}\Gamma(2\nu+2m+1)\frac{J_{2m+\nu+1/2}(\epsilon_j\alpha)}{(2\epsilon_j\alpha)^{\nu+1/2}},$$

$$h_j^-(\alpha) = -i\frac{4a_j\pi}{\Gamma(\nu+1/2)}$$

$$\times\sum_{m=0}^{\infty}\mu_{2m+1}^j\frac{(-1)^m}{(2m+1)!}\Gamma(2\nu+2m+2)\frac{J_{2m+1+\nu+1/2}(\epsilon_j\alpha)}{(2\epsilon_j\alpha)^{\nu+1/2}}. \tag{5}$$

where $J_\nu(x)$ are the Bessel functions, and $\Gamma(x)$ is the gamma function.

We see from representations (5) that as $\alpha\to\infty$, the functions $h_j^{\pm}(\alpha) \sim O(\alpha^{-(\nu+1)})$. Consequently, this order of attenuation of the Fourier transforms $h_j(\alpha)$ satisfies the condition at the edge for the functions $\{\mu_j(\eta_j)\}_{j=1}^N$.

Writing the system of two coupled integral equations for the functions $h_j(\alpha)$ with the limits of integration $(0, \infty)$, instead of the system of equations (3), and using (5), we easily see that the homogeneous equations in these systems are satisfied identically. To determine the unknown terms $\{\mu_m^j\}_{m=0}^{\infty}$, we substitute representations (5) into the inhomogeneous equations for the functions $\{h_j^{\pm}(\alpha)\}_{j=1}^N$. Introducing the quantity $\gamma(\alpha)$ from the formula

$$\sqrt{1-\alpha^2} = i\alpha[\gamma(\alpha)-1], \quad \gamma(\alpha) \underset{\alpha\to\infty}{=} O(\alpha^{-2}), \tag{6}$$

which corresponds to the separation of the operators that are generated by the left sides of the inhomogeneous equations in (3), into a principal (singular) part and completely continuous part, and making use of the discontinuous Sonin–Weber–Schafheitlien integrals,[8] we obtain the following system of coupled linear algebraic equations for determining the unknown terms $\{\mu_{2m}^j\}_{m=0}^{\infty}$ and $\{\mu_{2m+1}^j\}_{m=0}^{\infty}$:

$$\sum_{m=0}^{\infty}(-1)^m\mu_{2m}^j\beta_{\nu,2m}(C_{2k,2m}^{\nu}-Q_{2k,2m}^{\nu j}) = d_{2k,j}^{\nu}$$

$$+\sum_{q=1,q\neq j}^{N}\sum_{m=0}^{\infty}(-1)^m(\mu_{2m}^q\beta_{\nu,2m})P_{2m,}^{jq}$$

$$_{2k} - i\mu_{2m+1}^q\beta_{\nu,2m+1}P_{2m+1,2k}^{jq}), \tag{7}$$

$$j=1,2,\ldots,N, \quad k=0,1,2,\ldots$$

$$\sum_{m=0}^{\infty}(-1)^m\mu_{2m+1}^j\beta_{\nu,2m+1}(C_{2k+1,2m+1}^{\nu}-Q_{2k+1,2m+1}^{\nu j}) = d_{2k+1,j}^{\nu}$$

$$+\sum_{q=1,q\neq j}^{N}\sum_{m=0}(-1)^m(\mu_{2m}^q\beta_{\nu,2m}P_{2m,2k+1}^{jq}$$

$$-i\mu_{2m+1}^q\beta_{\nu,2m+1}P_{2m+1,2k+1}^{jq}),$$

$$j=1,2,\ldots,N, \quad k=0,1,2,\ldots \tag{8}$$

where

$$\beta_{\nu,m} = \frac{\Gamma(2\nu+m+1)}{m!};$$

$$C_{k,m}^{\nu} = \frac{\Gamma(2\nu)\Gamma(1/2(m+k)+1)}{\Gamma(\nu+1/2+1/2(k-m))\Gamma(\nu+1/2+1/2(m-k))\Gamma(1/2(m+k)+2\nu+1)}$$

$$Q_{k,m}^{\nu j} = \epsilon_j\left(\frac{2}{\epsilon_j}\right)^{2\nu}\int_0^{\infty}J_{k+\nu+1/2}(\epsilon_j\alpha)J_{m+\nu+1/2}(\epsilon_j\alpha)\gamma(\alpha)\frac{d\alpha}{\alpha^{2\nu}}.$$

$$P_{m,k}^{jq} = \frac{a_q}{a_j}\frac{\epsilon_j^2\cdot 2^{2\nu-1}}{(\epsilon_j\epsilon_q)^{\nu+1/2}}\int_{-\infty}^{\infty}A_{jq}(\alpha)J_{m+\nu+1/2}(\epsilon_q\alpha)J_{k+\nu+1/2}(\epsilon_j B_{jq}(\alpha))$$

$$\times\frac{e^{ikl_{jq}D_{jq}(\alpha)}}{[B_{jq}(\alpha)]^{\nu+1/2}}\frac{d\alpha}{\alpha^{\nu+1/2}},$$

$$d_{k,j}^{\nu} = (-1)^k i^{k+1}\epsilon_j 2^{2\nu-1}\left(\frac{2}{\epsilon_j}\right)^{\nu+1/2}\Gamma(\nu+1/2)$$

$$\times\frac{A_j(\alpha_0)}{[B_j(\alpha_0)]^{\nu+1/2}}J_{k+\nu+1/2}(\epsilon_j B_j(\alpha_0)), \quad e^{-ikr_{0j}D_j(\alpha)},$$

where

$$D_j(\alpha_0) = \alpha_0\cos\varphi_{0j} + \sqrt{1-\alpha_0^2}\sin\varphi_{0j},$$

It can be shown that the systems of linear algebraic equations (7) and (8) are uniquely soluble in the space of numerical sequences

$$l_2(\nu) \overset{\text{def}}{=} \left\{ \sum_{m=0}^{\infty} m^{2\nu} |\mu_m^j|^2 < \infty \right\}, \quad j = 1, 2, \ldots, N.$$

We note that the finite sum over the index q in the system of equations (7) and (8) describes the electrodynamic interaction of the j-th face with all the other faces.

As can be seen from (9), the matrix elements $C^{\nu}_{k,m}$ do not depend on the index j or on the frequency parameter ϵ_j. Here $\beta_{\nu,k} C^{\nu}_{k,k} = \Gamma(2\nu)/\Gamma^2(\nu + 1/2)$. This stems from the fact that the separation (6) corresponds to the isolation of the static part of the integral operator in the system of equations (3). Consideration of this property is very essential in the computer solution of the system of equations (7) and (8) by the reduction method. From this point of view, it is also very important that there is a simple recurrence relation for all matrix elements

$$L_{nm} = \frac{m-1}{n} (L_{m-1,n-1} + L_{m-1,n+1}) - L_{m-2,n},$$

where L_{nm} denotes any of the quantities $C^{\nu}_{k,m}$, $Q^{\nu j}_{k,m}$, or $P^{jq}_{m,k}$.

The problem of finding the scattered fields H_z^j is completely solved by the determination of the unknown terms $\{\mu^j_{2m}\}_{m=0}^{\infty}$ and $\{\mu^j_{2m+1}\}_{m=0}^{\infty}$ from the system of linear algebraic equations (7) and (8).

1) The case of E polarization is examined in a similar manner.

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A model for describing the development of the large-scale structure of the universe

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Astronomical data that have been obtained in recent years show that the spatial distribution of the galaxies has a distinctive structure, reminiscent of gigantic honeycombs of irregular shape. The inner regions of the cells are virtually free of bright galaxies, which are concentrated mainly in comparatively thin walls and in the edges of the cells.[1,2] The greatest density of galaxies is attained in rich clusters situated at the vertices of the cells; however the total number of galaxies, contained in these clusters does not exceed several percent. Such structure in the distribution of the density of matter is formed naturally in the nonlinear stage of the gravitational instability involving random (but smooth) initial perturbations.[3-6] Systems of collisionless[7] or agglutinating[8,9] particles moving inertially are other examples of systems in which cellular structure of this sort arises.

The approximate theory of Zel'dovich[3,5] describes the development of nonlinear gravitational instability up to the appearance of the first strongly flattened objects, "pancakes." According to this theory, the Eulerian coordinates of the particles vary according to the law

$$\mathbf{r}'(\mathbf{q}, t) = a(t)\,[\mathbf{q} - b(t)\mathbf{s}(\mathbf{q})], \tag{1}$$

where **q** are the Lagrangian coordinates of the particles; *a* and **b** describe the general expansion of the universe and the gravitational increase of the perturbations, respectively; and **s** takes into account the initial density perturbations. However, a solution of Eq. (1) reflects the evolution of the pancakes inadequately, since if they are formed from gas, the shock waves, which hinder the thickening of the pancake, must be taken into account after these pancakes are formed. In the specific case of collisionless particles, these shock waves are decelerated by the gravitational field of the massive pancake.

In this paper we propose a model equation for nonlinear diffusion

$$\mathbf{v}'_\tau + (\mathbf{v}\nabla)\mathbf{v} = \nu\nabla\mathbf{v}, \quad \mathbf{v}(\mathbf{r}, 0) = \mathbf{v}_0(\mathbf{r}), \tag{2}$$

whose solution, together with an equation of continuity

$$\rho'_\tau + \nabla(\rho\mathbf{v}) = 0, \quad \rho(\mathbf{r}, 0) = \rho_0, \tag{3}$$

describes the evolution of the fields of velocity $\mathbf{v}(\mathbf{r}, \tau)$ and density $\rho(\mathbf{r}, \tau)$, which qualitatively describes correctly the formation of the cellular structure.

Equation (2) is a three-dimensional analog of the Burgers equation that is well known in the theory of turbulence.[10-12] At $\nu = 0$ this equation describes the behavior of a cold beam of noninteracting particles; after the substitution of the variables

$$\mathbf{r} = \mathbf{r}'/a(t), \quad \tau = b(t), \quad \mathbf{v}_0(\mathbf{q}) = -\mathbf{s}(\mathbf{q}) \tag{4}$$

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