Wave diffraction by a plane strip of finite thickness

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The practical application of conducting screens in radio physics and electronics imposes more rigid requirements on the mathematical models. The model of an infinitesimally thin, perfectly conducting screen, which allows a significant simplification of the solution of the boundary-value problem, is used to investigate the electrodynamic properties of various obvstacles. Rejection of the model of an infinitesimally thin screen leads to qualitatively new difficulties in the construction of a mathematically rigorous solution of the diffraction problem, even in the case of a plane strip of finite thickness. Correct consideration of the screen thickness and establishment of the bounds of validity of the model of an infinitesimally thin screen are particularly pressing in investigations in the millimeter wave range.

1. Let a plane H-polarized wave[1)] $H_z^0 = e^{i(x\cos\theta_0+y\sin\theta_0)}$ be incident at an angle θ_0 to the X-axis on a perfectly conducting cylinder of rectangular cross section (see Fig. 1). The sides of the rectangle are $2a_1$ and $2a_2$ wide. We introduce the principal coordinate system x, y, with origin at the center of the cylinder, and the local coordinate systems x', y', with origins at the centers of the faces.

It is required to determine the scattered field. The hybrid method proposed in Refs. 1 and 2 is used to solve this diffraction problem. It is based on the idea of the method of moments and semi-inversion. First, following Refs. 1 and 2, we represent the functions $\{\mu_s\}_{s=1}^4$ for the surface current densities induced at the faces of the cylinder as a series over a system of base functions with an appropriate weight factor ensuring the specified behavior of the current near the edge. In the case under consideration, the functions $\{\mu_s\}_{s=1}^4$, on approaching the edges (the corners of the rectangle), tend to some constant as $O(\rho^{2/3})$, where $\rho \to 0$ is the distance to the edge. In order that this condition be satisfied, we seek the functions $\{\mu_s\}_{s=1}^4$ in the local coordinate systems x', y' in the form

$$\mu_s(\eta_s) = C_s(1-\eta_s) + C_{s+1}(1+\eta_s) + (1-\eta_s^2)^\nu \varphi_s(\eta_s), \tag{1}$$

where $\{C_s\}_{s=1}^4$ are unknown constants, $\{\eta_s = x_s/a_s\}_{s=1}^4 \in [-1, 1]$ is the normalized coordinate, $\nu = 2/3$, and $\{\varphi_s(\eta_s)\}_{s=1}^4$ are certain regular functions which belong to the space of functions $L_2(-1, 1; (1-\eta^2)^\nu)$, in which the scalar product is defined with the weight factor $(1-\eta^2)^\nu$. In this space, the functions $\{\varphi_s(\eta_s)\}_{s=1}^4$ can be explained in uniformly converging series in the Gegenbauer polynomials $\{C_n^{\nu+1/2}(\eta_s)\}_{n=0}^\infty$:

$$\varphi_s(\eta_s) = \sum_{n=0}^{\infty} \mu_n^s C_n^{\nu+1/2}(\eta_s), \quad \nu = 2/3, \tag{2}$$

where $\{\mu_n^s\}_{n=0}^\infty$ are unknown coefficients to be determined.

The continuity conditions at the edges, where $\eta_s = \pm 1$,

$$\mu_1(+1) = \mu_2(-1) = 2C_2, \ldots, \mu_4(+1) = \mu_1(-1) = 2C_1 \tag{3}$$

hold for the functions $\{\mu_s(\eta_s)\}_{s=1}^4$.

The diffraction field can then be represented as a superposition of the field scattered by each of the faces of the cylinder, i.e., $H_z^s = \sum_{s=1}^{4} H_z^s$.

The scattered fields $\{H_z^s\}_{s=1}^4$ can be determined in turn as a double-layer potential with density corresponding to the current density functions[3] $\mu_s(\eta_s)$:

$$H_z^s = \frac{i}{4}\int_{-1}^{1} \mu_s(\eta_s')\frac{\partial}{\partial\zeta_s} H_0^{(1)}\left(\epsilon_s\sqrt{(\eta_s-\eta_s')^2+\zeta_s^2}\right)d\eta_s', \tag{4}$$

where $\zeta_s = y_s/a_s$, $\epsilon_s = ka_s$ is the frequency parameter, and $H_0^{(1)}(x)$ is the zero-order Hankel function (the two dimensional Green's function of free space). We recall that the current function $\mu_s(\eta_s)$ is equal within a constant, to the z component of the total magnetic field at the surface of the cylinder.

In converting the Fourier transforms $\{\mu_s\}_{s=1}^4$, after subjecting the total field to the Neumann boundary condition at the surface of the cylinder, we obtain a coupled system of dual integral equations.[1,2] Making use of the Weber–Schafheitlien discontinuity integrals and partially inverting the operator of the problem, we obtain an infinite system of linear algebraic equations for $\{C_s\}_{s=1}^N$ and $\{\mu_n^s\}_{n=0}^\infty$

$$C_{l+1}[C_k^{(-l)} - d_k^{(-l)}] - C_l[C_k^{(+l)} - d_k^{(+l)}] + \sum_{m=0}^{\infty} x_m^{(l)}[C_{km}^{(\nu)} - d_{km}^{(\nu)}]$$
$$= f_k^{(l)} - \sum_{q=1, q\neq l}^{4}\left\{C_{q+1}P_k^{-l,q} - C_q P_k^{+l,q} + \sum_{m=0}^{\infty} x_m^{(q)} P_{km}^{lq}\right\},$$
$$l = 1, 2, 3, 4, \tag{5}$$

where we have introduced the notation

$$x_m^{(l)} = (-i)^m \mu_m^{(l)} \beta_m^\nu; \quad \beta_m^\nu = \Gamma(m+2\nu+1)/\Gamma(m+1);$$

$$C_k^{(\pm l)} = \frac{i}{\pi} k_\nu^{ll} \int_{-\infty}^{\infty} \frac{|\alpha|}{\alpha} K_l(\pm\alpha) J_{k+\nu+1/2}(\epsilon_l\alpha)\frac{d\alpha}{\alpha^{\nu+1/2}};$$

$$k_\nu(\epsilon_l) = 2\left(\frac{2}{\epsilon_l}\right)^{2\nu-1}\frac{\Gamma^2(\nu+1/2)}{\Gamma(2\nu)};$$

$$d_k^{(\pm l)} = \frac{i}{\pi} k_\nu^{ll} \int_{-\infty}^{\infty} \frac{|\alpha|}{\alpha}\gamma(\alpha) K_l(\pm\alpha) J_{k+\nu+1/2}(\epsilon_l\alpha)\frac{d\alpha}{\alpha^{\nu+1/2}};$$

$$k_\nu^{lq} = 2(2\epsilon_q)^{\nu-1/2}\Gamma(\nu+1/2)k_\nu(\epsilon_l);$$

$$C_{km}^{(\nu)} = \frac{\Gamma^2(\nu+1/2)\Gamma((k+m)/2)+1)}{\Gamma(\nu+(1+k-m)/2)\Gamma(\nu+(1+m-k)/2)\Gamma((k+m)/2+2\nu+1)}$$
$$\times[1+(-1)^{k+m}];$$

$$Q_{km}^{(\nu)} = k_\nu(\epsilon_l)\int_0^\infty \gamma(\alpha) J_{k+\nu+1/2}(\epsilon_l\alpha) J_{m+\nu+1/2}(\epsilon_l\alpha)\frac{d\alpha}{\alpha^{2\nu}}[1+(-1)^{k+m}];$$

$$P_k^{\pm jq} = \frac{i}{\pi} k_\nu^{jq}\left(\frac{\epsilon_j}{\epsilon_q}\right)^{\nu-1/2} \int_{-\infty}^{\infty} A_{jq}(\alpha) K_q(\pm\alpha) \times \frac{e^{ikl_{jq}D_{jq}(\alpha)}}{[B_{jq}(\alpha)]^{\nu+1/2}} J_{k+\nu+1/2}(\epsilon_j B_{jq}(\alpha)) \frac{d\alpha}{\alpha};$$

$$P_{km}^{jq} = k_\nu(\epsilon_j)\left(\frac{\epsilon_j}{\epsilon_q}\right)^{\nu-1/2} \int_{-\infty}^{\infty} A_{jq}(\alpha) J_{m+\nu+1/2}(\epsilon_q \alpha) \times J_{k+\nu+1/2}(\epsilon_j B_{jq}(\alpha)) \frac{e^{ikl_{jq}D_{jq}(\alpha)}}{[\alpha B_{jq}(\alpha)]^{\nu+1/2}}\, d\alpha;$$

$$f_k^{(j)} = k_\nu(\epsilon_j) \cdot 2^{\nu+1/2}\Gamma(\nu+1/2) e^{ikr_{s,j}D_j(\alpha_0)} A_j(\alpha_0) \frac{J_{k+\nu+1/2}(\epsilon_j B_j(\alpha_0))}{[B_j(\alpha_0)]^{\nu+1/2}};$$

$$K_j(\pm\alpha) = e^{\pm i\epsilon_j\alpha} - \frac{\sin(\epsilon_j\alpha)}{\epsilon_j\alpha}; \qquad (6)$$

$\Gamma(x)$ is the gamma function, $J_\nu(x)$ is the Bessel function, and the quantities $A_{jq}(\alpha)$, $B_{jq}(\alpha)$, $D_{jq}(\alpha)$, $A_j(\alpha_0)$, $B_j(\alpha_0)$, and $D_j(\alpha_0)$ were calculated in Refs. 1 and 2.

To determine the coefficients $\{C_j\}_{j=1}^{4}$ in the system of linear algebraic equations, we use the additional relations

$$e^{-i\epsilon_j B_j(\alpha_0) + ikr_{s,j}D_j(\alpha_0)} + \sum_{q=1,\, q\neq j,\, j+1}^{4} \left\{ \sum_{m=0}^{\infty} x_m^{(q)} E_m^{jq} + i(C_{q+1}E_{jq}^{(-)} - C_q E_{jq}^{(+)}) \right\} = -C_j, \qquad (7)$$

$$j = 1, 2, 3, 4,$$

which follow from the definition of the current functions at the faces of the cylinder (1) and from the conditions at the edge (3). The quantities $\{E_m^{jq}\}_{m=0}^{\infty}$ and $\{E_{jq}^{(\pm)}\}_{j\neq q=1}^{4}$ are defined as

$$E_{jq}^{(\pm)} = \frac{1}{\pi}\int_{-\infty}^{\infty} K_q(\pm\alpha) e^{-i\epsilon_j B_{jq}(\alpha) + ikl_{jq}D_{jq}(\alpha)} \frac{d\alpha}{\alpha},$$

$$E_m^{jq} = \frac{\epsilon_q}{(2\epsilon_q)^{\nu+1/2}\Gamma(\nu+1/2)} \int_{-\infty}^{\infty} J_{m+\nu+1/2}(\epsilon_q\alpha) \times e^{-i\epsilon_j B_{jq}(\alpha) + ikl_{jq}D_{jq}(\alpha)} \frac{d\alpha}{\alpha^{\nu+1/2}}.$$

Using the condition that the energy in any finite volume of space is bounded, we can show that the unknowns $\{\mu_n^{(j)}\}_{n=0}^{\infty}$ belong to the class of numerical sequences

$$l_2(\nu) = \left\{\mu_n^j : \sum_{n=0}^{\infty} |\mu_n^{(j)}|^2 \beta_n^\nu < \infty, \qquad \beta_n^\nu \underset{n\to\infty}{=} O(n^{2\nu})\right\}. \qquad (8)$$

Upon switching to the new unknowns $y_n^{(j)} = \mu_n^{(j)}\sqrt{\beta_n^\nu}$, which belong to the class of numerical sequences l_2, according to (8), we can prove that the infinite system of linear algebraic equations for the unknowns $\{y_n^{(j)}\}_{n=0}^{\infty}$ is the Fredholm equation of the second kind, with completely continuous operator in l_2. Consequently, an approximate solution of this equation can be obtained to any prespecified accuracy by the reduction method.

2. An effective solution of the problem of wave diffraction by a plane strip in the approximation of an infinitesimally thin screen can also be obtained by this approach. Since the current density of the strip in this case is defined as a sudden change in the H_z component of the magnetic field, it tends to zero as $O(\rho^{1/2})$, upon

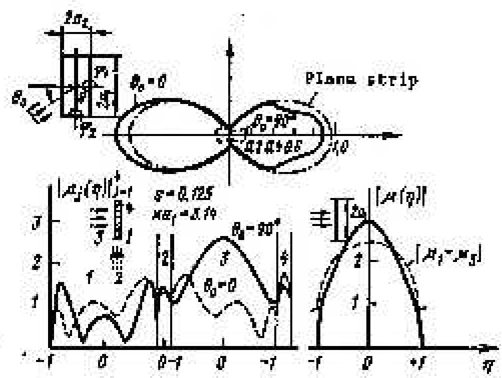

FIG. 1

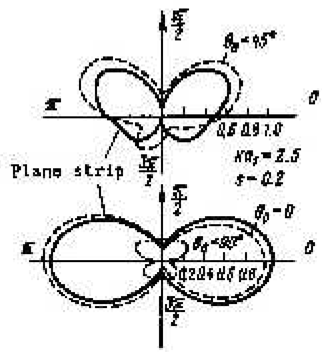

FIG. 2

approaching the edge, where $\rho \to 0$ (Ref. 3). For an infinitesimally thin strip, the current density $\mu(\eta)$ is therefore expanded in a series:

$$\mu(\eta) = (1-\eta^2)^{1/2} \sum_{n=0}^{\infty} x_n U_n(\eta), \qquad (9)$$

where $U_n(\eta)$ are the Chebyshev polynomials of the second kind. One can show that the unknown coefficients $\{x_n\}_{n=0}^{\infty}$ can be determined from the solution of the infinite system of linear algebraic equations of the second kind:

$$x_k - \sum_{n=0}^{\infty} d_{kn}^{(1/2)} x_n = f_k, \quad k = 0, 1, 2, \ldots, \qquad (10)$$

where $\{d_{kn}^{(1/2)}\}_{k,n=0}^{\infty}$ are determined by relations (6), in which one sets $\nu = 1/2$.

3. A set of sufficiently effective algorithms for calculation of the matrix elements of the system of linear algebraic equations (5), the matrix of the additional system of four equations (7), and for calculation of the surface current densities and diffraction patterns of the scattered field was constructed for numerical solution of this problem.

Figures 1-3 show the diffraction patterns of the fields and the surface current distributions for an infinitesimally thin strip and for a cylinder with a rectangular cross section for different values of the frequency parameter, angle of incidence, and the parameter $s = a_2/a_1$, which gives the "thickness" of the cylinder. The following conclusions can be drawn from analysis of the results shown in these figures.

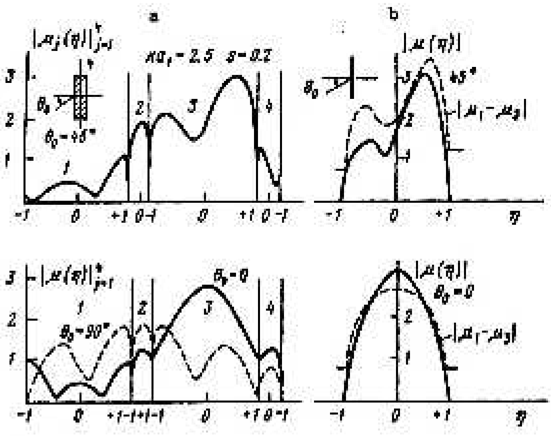

FIG. 3

For normal incidence of the wave ($\theta_0 = 0°$; Figs. 1 and 2) upon the wide face of the rectangular cylinder with thickness $0.1 \le s \le 0.5$, the diffraction pattern of the field is close to that for the plane strip. The difference is seen in that the field amplitude in the far zone on the side of the illuminated face slightly exceeds the field amplitude in the shadow region, and the diffraction pattern does not vanish in the directions $\phi = 90°$ and $\phi = 270°$ (the side faces contribute to the diffraction pattern). However, if a whole number of half-wavelengths fits across the wide face of the rectangular cylinder, the difference between the diffraction patterns of the plane strip and cylinder decreases.

In the case of oblique incidence (Figs. 2 and 3) on a thin rectangular cylinder (on a plane strip of finite thickness), the diffraction pattern becomes asymmetric as compared to the diffraction pattern of an infinitesimally thin strip. The thickness of the strip has a notable effect on the diffraction pattern and on the current density distribution. The current amplitude on the side face is large, because the function which gives the current density on the faces of the cylinder is piecewise continuous in the case of diffraction of the H-polarized wave. In other words, the surface current, by passing the edge, in this case effectively penetrates the shadow zone through the side faces. For a clearer comparison of these results, we used graphs of the current density distribution on an infinitesimally thin strip – the discontinuity of the field component H_z – to show the difference in the surface current densities induced on the sides of the strip of finite thickness $|\mu_1(\eta) - \mu_3(\eta)|$.

In many problems of microwave technology it becomes necessary to evaluate ohmic losses in metal screens. These losses are proportional to the integral of the square of the current at the surface of the screen. It is known that to evaluate this quantity on the basis of the perturbation theory (in the case of a strong skin effect in metals) for H polarization, we can use the results obtained by rigorous methods for infinitesimally thin, perfectly conducting screens.[4] Our investigation of the current distribution along the surface of a strip of finite thickness has shown that the use of the model of an infinitesimally thin screen can result in a significant error in determining ohmic losses.

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[1])The time dependence $e^{-i\omega t}$ is omitted everywhere.

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