Wave diffraction by intersecting circular cylindrical bodies

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A rigorous effective numerical method is proposed for the solution of the problem of wave diffraction by cylindrical bodies formed from parts of intersecting circular cylinders. For simplicity, we shall examine the case of two intersecting cylinders. No fundamental difficulties are encountered in the generalization to cylindrical bodies formed by the intersection of several cylinders.

The proposed method involves finding the scattered field from each side of such a cylinder (the total scattered field is the superposition of these fields), which reduces to the solution of a system of coupled conjugate summator equations for the Fourier coefficients of the functions describing the surface current density on them. The method of moments[1] in combination with the semi-inversion method[2] is applied to the solution of these equations. As a result, the problem leads to a coupled infinite system of linear algebraic equations, which are solved by the reduction method. The unknown terms in these equations are the coefficients in terms of which the surface current densities are represented by the complete system of Gegenbauer polynomials with a weight factor taking into account the behavior of the current functions at the edges.

1. Let a plane H-polarized electromagnetic wave[1)] $H_z^0 = e^{ik(\alpha_0 x - \beta_0 y)}$ ($k = 2\pi/\lambda$, λ is the wavelength, $\alpha_0 = \cos\vartheta_0$, $\beta_0 = \sin\vartheta_0$, the time dependence is $e^{-i\omega t}$) be incident upon the cylindrical body under consideration (see Fig. 1) at an angle ϑ_0 from the side $y < 0$. It is required to determine the scattered field which must satisfy the wave equation outside the cylinder, the radiation condition at infinity, the Neumann boundary condition at the surface of the cylinder, and the condition of finite energy in any bounded volume of space (the Meichsner condition at the edges).

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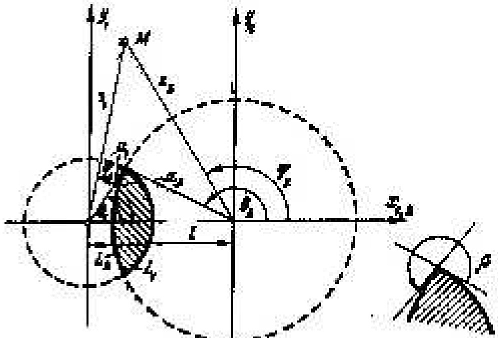

FIG. 1

We shall represent the total field as $H_z = H_z^0 + \sum_{s=1}^{2} H_z^s$, where H_z^s describes the scattered field produced by the surface current on the part of the cylindrical surface with number s. We shall seek these fields in local cylindrical coordinate systems coupled to each of the intersecting cylinders in the form[2]

$$H_z^s = \sum_{m=-\infty}^{\infty} \mu_m^s \begin{Bmatrix} J'_m(\epsilon_s) H_m^{(1)}(kr_s) \\ J_m(kr_s) H_m^{(1)'}(\epsilon_s) \end{Bmatrix} e^{im\varphi_s}, \quad \begin{matrix} r_s > a_s, \\ r_s < a_s, \end{matrix} \tag{1}$$

where the unknowns $\{\mu_m^s\}_{m=-\infty}^{\infty}$ are the Fourier coefficients of the functions $\mu_s(\varphi_s) = \frac{2}{i\epsilon_s} \sum_{m=-\infty}^{\infty} \mu_m^s e^{im\varphi_s}$, $\varphi_s \in (\theta_s, 2\pi - \theta_s)$, which describe the surface current density on the cylindrical surfaces with angular size $\{2\theta_s\}_{s=1}^{2}$, $J_m(x)$ and $H_m^{(1)}(x)$ are the Bessel and Hankel functions, respectively, $\epsilon_s = ka_s$, and a_s is the radius of the s-th cylinder.

To find the unknowns $\{\mu_m^s\}_{m=-\infty}^{\infty}$, we subject the total field H_z to the Neumann boundary condition at the surface of the cylindrical body under consideration:

$$\frac{\partial}{\partial n}\left(H_z^0 + \sum_{s=1}^{2} H_z^s\right)\Big|_L = 0, \quad L = \bigcup_{s=1}^{2} L_s^1, \tag{2}$$

where L_s^1 is the contour of the part of the cylindrical body with number s (see Fig. 1) and the normal n is directed along the radius vector r.

In order to satisfy Eq. (2), the field $\{H_z^j\}_{j=1}^{2}$ must be written in the systems of coordinates coupled to the cylinders with numbers s = 2, 1, respectively. Then, on the basis of the addition theorem for the cylindrical functions, to find the unknowns $\{\mu_m^s\}_{m=-\infty}^{\infty}$ we obtain from Eq. (2) the coupled system of conjugate summator equations

$$\sum_{m=-\infty}^{\infty} (-1)^m \mu_m^j \gamma_m^j e^{im\alpha_j \zeta_j} = -\sum_{m=-\infty}^{\infty} (-i)^m J'_m(\epsilon_j) e^{im(\alpha_j \zeta_j - \theta_j)}$$
$$- \sum_{q=1, q\neq j}^{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^n \mu_m^q J'_n(\epsilon_j) P_{mn}^{qj} e^{in\alpha_j \zeta_j},$$
$$|\zeta_j| < 1, \quad j = 1, 2, \qquad \sum_{m=-\infty}^{\infty} (-1)^m \mu_m^j e^{im\alpha_j \zeta_j} = 0, \quad |\zeta_j| > 1, \tag{3}$$

where $\alpha_j = \pi - \theta_j$, $\zeta_j = \frac{\varphi_j - \pi}{\alpha_j}$; $\gamma_m^j = J'_m(\epsilon_j) H_m^{(1)'}(\epsilon_j)$, l is the distance between the centers of the cylinders, and

$$P_{mn}^{lq} = \begin{bmatrix} P_{mn}^{12} \\ P_{mn}^{12} \\ P_{mn}^{21} \\ P_{mn}^{21} \end{bmatrix} = \begin{Bmatrix} H_{m-n}^{(1)}(kl) J'_m(\epsilon_1) J'_n(\epsilon_2) \\ J_{m-n}(kl) J'_m(\epsilon_1) H_n^{(1)'}(\epsilon_2) \\ H_{n-m}^{(1)}(kl) J'_m(\epsilon_2) J'_n(\epsilon_1) \\ J_{n-m}(kl) J'_m(\epsilon_2) H_n^{(1)'}(\epsilon_1) \end{Bmatrix}, \quad \begin{matrix} l > a_2, \\ l < a_2, \\ l > a_1, \\ l < a_1. \end{matrix}$$

We note that the homogeneous equations in system (3) are obtained by the extension of the function $\mu_j(\zeta_j)$ by zero outside the interval $\zeta_j \in [-1, 1]$.

The unknown Fourier coefficients $\{\mu_m^j\}_{m=-\infty}^{\infty}$ must satisfy, besides the system of equations (3), the relations $\sum_{m=-\infty}^{\infty} m|\mu_m^j|^2 < \infty$, j = 1, 2, which follow from the condition that the energy of the scattered fields be finite in any bounded part of space. It can be shown[2] that the solution of the system of equations (3) that satisfies this condition exists and is unique.

2. The solution of the system of conjugate summator equations can be obtained by representing the functions $\mu_j(\zeta_j)$ as uniformly convergent series in the complete and orthogonal system of Gegenbauer polynomials[3] $\{C_m^{\nu}(\zeta_j)\}_{m=0}^{\infty}$ with a weight factor taking into account the behavior of the current function at the ends of the interval $\zeta_j \in [-1, 1]$. For the H-polarization under consideration, this representation has the form

$$\mu_j(\zeta_j) = (1 - \zeta_j^2)^{\nu} \sum_{m=0}^{\infty} x_m^j C_m^{\nu+1/2}(\zeta_j), \tag{4}$$

where $\{x_m^j\}_{m=0}^{\infty}$ are the new unknown coefficients, $\nu = \pi/\beta$, and β is the angle between the two tangents passing through the point of intersection of the cylinders (see Fig. 1).

By representing the functions $\{\mu_j(\zeta_j)\}_{j=1}^{2}$ and their Fourier coefficients as the sums of even and odd terms $\mu_{\pm m}^j = (1/2)[\mu_m^{+j} \pm \mu_m^{-j}]$ and using (4), we can obtain representations for $\mu_m^{\pm j}$ a uniformly convergent series in the Bessel functions:

$$\mu_m^{+j} = \frac{2i\pi\epsilon_j\alpha_j(-1)^m}{\Gamma(\nu + 1/2)} \sum_{n=0}^{\infty} x_{2n}^j \frac{(-1)^n}{(2n)!}$$
$$\times \Gamma(2\nu + 1 + 2n) \frac{J_{2n+\nu+1/2}(m\alpha_j)}{(2m\alpha_j)^{\nu+1/2}},$$
$$\mu_m^{-j} = \frac{2\pi\epsilon_j\alpha_j(-1)^m}{\Gamma(\nu + 1/2)} \sum_{n=0}^{\infty} x_{2n+1}^j \frac{(-1)^n}{(2n+1)!}$$
$$\times \Gamma(2\nu + 2 + 2n) \frac{J_{2n+1+\nu+1/2}(m\alpha_j)}{(2m\alpha_j)^{\nu+1/2}}, \tag{5}$$

where $J_\nu(x)$ are the Bessel functions, and $\Gamma(x)$ is the gamma function.

It follows from (5) that the coefficients $\mu_m^j \sim O(m^{-(\nu+1)})$ as $m \to \infty$; i.e., this order of decrease of the Fourier coefficients of the functions $\mu_j(\zeta_j)$ ensures satisfaction of the Meichsner condition at the edges for the functions themselves. Moreover, it also follows from these representations with use of the discontinuous Weber–Schafheitlin intervals[3] that in the system of conjugate summator equations in the coefficients $\mu_m^{\pm j}$ the homogeneous equations are satisfied identically.

Let us introduce the quantities $\{\delta_m^j\}_{m=1}^{\infty}$, which are expressed in terms of γ_m^j in the following manner:

$$\delta_m^j = 1 + i\pi\epsilon_j^2 \frac{\gamma_m^j}{m}, \quad m \neq 0; \quad \delta_m^j \underset{m\to\infty}{=} O\left(\frac{1}{m^2}\right). \tag{6}$$

This corresponds to the separation of the operators generated by the left sides of the inhomogeneous equations in the system (3) into a principal (singular) part and completely continuous parts. Furthermore, by substituting representations (5) for the Fourier coefficients into these equations and using the completeness of the Gegenbauer polynomials in the interval [−1, 1] we obtain for the unknowns $\{x_m^j\}_{m=0}^{\infty}$ the infinite system of coupled linear algebraic equations

$$\sum_{n=0}^{\infty}(-1)^n x_{2n}^j (d_{2k,2n}^{\nu j} - N_{2k,2n}^{\nu j})\beta_{\nu,2n} = \Gamma_{2k,j}^{+\nu} + x_0^j A_j^\nu \delta_{k0}$$
$$+ \sum_{q=1, q\neq j}^{2} \sum_{p=0}^{\infty} (-1)^p (x_{2p}^q \beta_{\nu,2p} M_{2k,2p}^{+jq}$$
$$- i x_{2p+1}^q \beta_{\nu,2p+1} M_{2k,2p+1}^{-jq}), \tag{7}$$

$$\sum_{n=0}^{\infty}(-1)^n x_{2n+1}^j (d_{2k+1,2n+1}^{\nu j} - N_{2k+1,2n+1}^{\nu j})\beta_{\nu,2n+1} = \Gamma_{2k+1,j}^{-\nu}$$
$$+ i \sum_{q=1, q\neq j}^{2} \sum_{p=0}^{\infty} (-1)^p (x_{2p}^q \beta_{\nu,2p} M_{2k+1,2p}^{+jq}$$
$$- i x_{2p+1}^q \beta_{\nu,2p+1} M_{2k+1,2p+1}^{-jq}), \tag{8}$$

where

$$\beta_{\nu,m} = \frac{\Gamma(2\nu+m+1)}{m!}; \quad d_{k,n}^{\nu j} = \frac{\sqrt{\pi}\,\Gamma(\nu+½)}{\alpha_j^{2\nu-1}\Gamma(\nu)} \sum_{m=1}^{\infty} \frac{1}{m^{2\nu}}$$
$$\times J_{k+\nu+½}(m\alpha_j) J_{n+\nu+½}(m\alpha_j),$$

$$N_{k,n}^{\nu j} = \frac{\sqrt{\pi}\,\Gamma(\nu+½)}{\alpha_j^{2\nu-1}\Gamma(\nu)} \sum_{m=1}^{\infty} \frac{\delta_m^j}{m^{2\nu}} J_{k+\nu+½}(m\alpha_j) J_{n+\nu+½}(m\alpha_j);$$

$$A_j^\nu = \frac{i\pi^{3/2}\epsilon_j^2 \alpha_j^2 \Gamma(\nu+½)\Gamma(2\nu+1)\gamma_0^j}{2^{2\nu+2}\Gamma^2(\nu+3/2)\Gamma(\nu)},$$

$$M_{k,p}^{\pm jq} = \frac{i\pi^{3/2}\epsilon_j\epsilon_q}{(\alpha_j\alpha_q)^{\nu+½}} \frac{\Gamma(\nu+½)}{\Gamma(\nu)} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \delta_m \frac{(-1)^{m+n}}{(mn)^{\nu+½}} (P_{mn}^{jq}$$
$$\pm P_{-mn}^{jq}) J_{k+\nu+½}(n\alpha_j) J_{p+\nu+½}(m\alpha_j),$$

$$\Gamma_{k,j}^{\pm\nu} = \sqrt{\pi}\,\alpha_j\epsilon_j \left(\frac{2}{\alpha_j}\right)^{\nu+½} \frac{\Gamma(\nu+½)}{\Gamma(\nu)} \sum_{m=0}^{\infty} \frac{(-i)^m}{m^{\nu+½}} J_m'(\epsilon_j) J_{k+\nu+½}(m\alpha_j)$$
$$\times \begin{Bmatrix} \cos(m\beta_j) \\ \sin(m\beta_j) \end{Bmatrix}; \quad \delta_m = \begin{cases} ½, & m = 0, \\ 1, & m \neq 0. \end{cases}$$

We note that the following representation can be obtained for the matrix elements $d_{k,n}^{\nu j}$, which are independent of the frequency parameter ϵ_j and are represented as weakly converging series, as in Refs. 4 and 5:

$$d_{k,n}^{\nu j} = d_{k,n}^{1\nu} + d_{k,nj}^{2\nu}$$
$$= \frac{2\Gamma^2(\nu+½)\Gamma(½(k+n)+2)}{(k+n+2)\Gamma(½(k+n)+2\nu+1)\Gamma(½+\nu+½(k-n))\Gamma(½+\nu-½(k-n))}$$
$$+ \sum_{s=1}^{\infty} D_{kns}^{\nu}\alpha_j^{2s}. \tag{9}$$

Here the $d_{k,n}^{1\nu}$ do not depend on the parameter α_j, and D_{kns}^{ν} decreases so rapidly with increasing indices that it is sufficient for practical calculations to retain the first several terms in the series in these quantities. It is seen from (9) that $d_{k,k}^{1\nu}\beta_\nu$, k=1, i.e., the systems of equations (7) and (8) are second-order equations.

We can show that the systems of linear algebraic equations [Eqs. (7) and (8)] are soluble in a unique manner by the reduction method in the space of numerical sequences $l_2(\nu) \overset{\text{def}}{=} \sum_{m=0}^{\infty} m^{2\nu}|x_m^j|^2 < \infty$, $j = 1, 2$. There is a simple recurrence relation for all the matrix elements:

$$L_{n,m} = \frac{m-1}{n}(L_{m-1,n-1} + L_{m-1,n+1}) - L_{m-2,n}, \tag{10}$$

where $L_{n,m}$ denotes any of the quantities $d_{k,n}^{\nu j}$, $N_{k,n}^{\nu j}$, and $M_{k,p}^{\pm jq}$. The use of relations (9) and (10) in the computer solution of the system of equations (7), (8) by the reduction method significantly simplifies the computation.

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1)The case of E-polarization is examined in a similar manner.

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