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BIFURCATIONS OF ELASTIC ROTORS IN JOURNAL BEARINGS

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ABSTRACT

The model of nonlinear vibrations of one disk rotor supported by two journal bearing is obtained. The fluid film of journal bearing is described by the Reynolds' equation. Shaw-Pierre nonlinear modes, harmonic balance method and continuation technique are used to study the rotor dynamics. Self-sustained vibrations of the rotor take place at rotor angular velocity much lower then the angular velocity of Hopf bifurcation. These self vibrations occur due to saddle-node bifurcations.

INTRODUCTION

Self-sustained vibrations of rotors take place due to influence of carrier fluid film on the rotor motions. Self-sustained vibrations lead up to failure of several turbines [1]. Now the modern methods of nonlinear dynamics are used to analyze the self-sustained vibrations of rotors [2]. Akers [3] analyzed the pressure of a fluid film of the journal bearing. Poznjak [4] analytically describe the pressure of fluid film as a function of general coordinates, velocities and acceleration. Olimpiev [5] obtained the asymptotic solution of the Reynolds' equation using the variational approach. Karintsev, Shul'genko [6] obtained the model of pressure in fluid film of short journal bearings. The vibrations of symmetric rotor supported by short journal bearings are treated in [7]. Ovcharova, Goloskokov [8] analyzed the rotor forced vibrations accounting short journal bearings. They described the shaft dynamics by three discrete masses. Muszynska [9, 10] considers the symmetric rotor with one journal bearing. The mathematical model of rotor dynamics based on experimental data is treated.

In this paper one disk rotor supported by two journal bearings are considered. Shaw-Pierre nonlinear modes are used to study self-sustained vibrations of rotor.

1. PROBLEM FORMULATION AND MAIN EQUATIONS

It is assumed that the disk is rigid and the shaft is elastic (Fig.1). The shaft is supported by two short journal bearings.

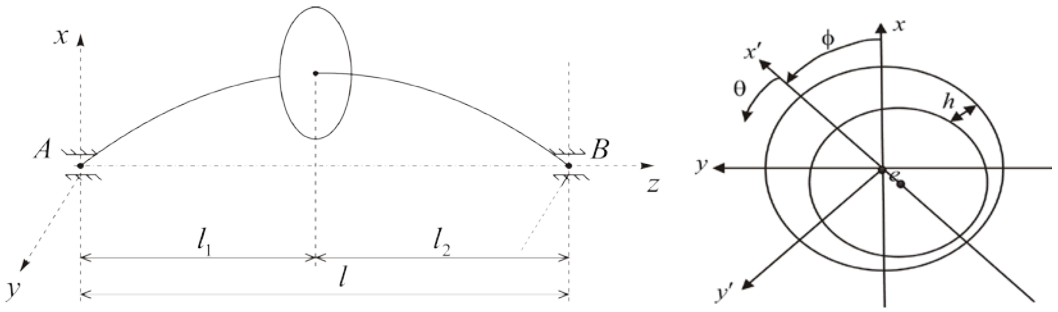

Fig. 1 Sketch of one disk rotor

During disk vibrations, the parts of the shaft in journal bearings A and B are vibrated too. The vibrations of the journals A and B (Fig.1) is described by the general coordinates (x_1, y_1) and

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(x_2, y_2), respectively. Two journal bearings are identical. Forces of carrier fluid film occur due to journal motions. The projections of these forces on the axes x and y are denoted by $F_x(x_i; y_i)$; $F_y(x_i; y_i)$; $i = \overline{1,2}$. The rotor is rotated with constant angular velocity Ω about z axes; the angular velocity of the disk has the following form:

$$\begin{aligned}
&\vec{\omega} = \omega_1 \vec{e}_1^{\,(3)} + \omega_2 \vec{e}_2^{\,(3)} + \omega_3 \vec{e}_3^{\,(3)} \\
&\omega_1 = \dot{\theta}_1 \cos\theta_2 \cos\theta_3 + \dot{\theta}_2 \sin\theta_3 \\
&\omega_2 = \dot{\theta}_2 \cos\theta_3 - \dot{\theta}_1 \cos\theta_2 \sin\theta_3 \\
&\omega_3 = \dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2
\end{aligned} \tag{1}$$

Then angular velocity of the rotor has following form: $\Omega = \dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2$. The kinetic energy of the disk are the following form:

$$T = \frac{I_e}{2}\left(\dot{\theta}_2^2 + \dot{\theta}_1^2 \cos^2\theta_2\right) + \frac{I_p}{2}\left(\dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2\right)^2 + \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \tag{2}$$

where x, y are displacements of the point, when the axis of the shaft and the disk are intersected; I_e, I_p are diametrical and polar moment of inertia of the rotor, respectively. The shaft potential energy has the following form:

$$\begin{aligned}
\Pi = &\frac{1}{2} c_{11}\left[(x - \varsigma_1 x_2 - \varsigma_2 x_1)^2 + (y - \varsigma_2 y_1 - \varsigma_1 y_2)^2\right] + \\
&+ \frac{1}{2} c_{22}\left[\left(\theta_2 - \frac{x_2 - x_1}{l}\right)^2 + \left(\theta_1 + \frac{y_2 - y_1}{l}\right)^2\right] + \\
&+ c_{12}\left[(x - \varsigma_1 x_2 - \varsigma_2 x_1)\left(\theta_2 - \frac{x_2 - x_1}{l}\right) - (y - \varsigma_2 y_1 - \varsigma_1 y_2)\left(\theta_1 + \frac{y_2 - y_1}{l}\right)\right]
\end{aligned} \tag{3}$$

where c_{11}, c_{22}, c_{12} are elements of stiffness matrix, $\varsigma_1 = \frac{l_1}{l}; \varsigma_2 = \frac{l_2}{l}$.

The equations of system motions with respect to the general coordinates $(x, y, \theta_1, \theta_2, x_1, y_1, x_2, y_2)$ have the following form:

$$\begin{aligned}
&m\ddot{x} + c_{11}(x - \varsigma_1 x_2 - \varsigma_2 x_1) + c_{12}\left(\theta_2 - \frac{x_2 - x_1}{l}\right) = -m g \\
&m\ddot{y} + c_{11}(y - \varsigma_1 y_2 - \varsigma_2 y_1) - c_{12}\left(\theta_1 + \frac{y_2 - y_1}{l}\right) = 0 \\
&I_e \ddot{\theta}_1 \cos^2\theta_2 - I_e \dot{\theta}_1 \dot{\theta}_2 \sin 2\theta_2 + I_p \ddot{\theta}_3 \sin\theta_2 + I_p \Omega \dot{\theta}_2 \cos\theta_2 - \frac{I_p}{2} \dot{\theta}_1 \dot{\theta}_2 \sin 2\theta_2 + \\
&+ I_p \ddot{\theta}_1 \sin^2\theta_2 + I_p \dot{\theta}_1 \dot{\theta}_2 \sin 2\theta_2 + c_{22}\left(\theta_1 + \frac{y_2 - y_1}{l}\right) - c_{12}(y - \varsigma_2 y_1 - \varsigma_1 y_2) = 0 \\
&I_e \ddot{\theta}_2 + \frac{I_e}{2} \dot{\theta}_1^2 \sin(2\theta_2) - I_p \Omega \dot{\theta}_1 \cos\theta_2 + c_{22}\left(\theta_2 - \frac{x_2 - x_1}{l}\right) + c_{12}(x - \varsigma_1 x_2 - \varsigma_2 x_1) = 0 \\
&\left(\frac{c_{12}}{l} - \varsigma_2 c_{11}\right)(x - \varsigma_1 x_2 - \varsigma_2 x_1) + \left(\frac{c_{22}}{l} - \varsigma_2 c_{12}\right)\left(\theta_2 - \frac{x_2 - x_1}{l}\right) = F_x(x_1, y_1)
\end{aligned} \tag{4}$$

$$\left(\frac{c_{12}}{l}-\varsigma_2 c_{11}\right)(y-\varsigma_1 y_2-\varsigma_2 y_1)+\left(\varsigma_2 c_{12}-\frac{c_{22}}{l}\right)\left(\theta_1+\frac{y_2-y_1}{l}\right)=F_y(x_1,y_1)$$
$$\left(\varsigma_1 c_{11}+\frac{c_{12}}{l}\right)(x-\varsigma_1 x_2-\varsigma_2 x_1)+\left(\frac{c_{22}}{l}+\varsigma_1 c_{12}\right)\left(\theta_2-\frac{x_2-x_1}{l}\right)=-F_x(x_2,y_2)$$
$$\left(\frac{c_{22}}{l}+\varsigma_1 c_{12}\right)\left(\theta_1+\frac{y_2-y_1}{l}\right)-\left(\varsigma_1 c_{11}+\frac{c_{12}}{l}\right)(y-\varsigma_2 y_1-\varsigma_1 y_2)=F_y(x_2,y_2)$$

Under the action of the gravity, the rotor takes up some equilibrium positions, which defines by the following values of the general coordinates: $(\bar{x},\bar{y},\bar{\theta}_1,\bar{\theta}_2,\bar{x}_1,\bar{y}_1,\bar{x}_2,\bar{y}_2)$. The rotor vibrations with respect to this equilibrium position are analyzed. Then the following change of the variables is used:

$$\begin{aligned}&(x,y,\theta_1,\theta_2,x_1,y_1,x_2,y_2)\rightarrow\\&\rightarrow(\bar{x}+x,\ \bar{y}+y,\ \bar{\theta}_1+\theta_1,\ \bar{\theta}_2+\theta_2,\ \bar{x}_1+x_1,\ \bar{y}_1+y_1,\ \bar{x}_2+x_2,\ \bar{y}_2+y_2)\end{aligned} \tag{5}$$

As a result the following dynamical system is derived:

$$\begin{aligned}m\ddot{x}&=R_X^{(1)} \quad I_e\ddot{\theta}_2-I_p\Omega\dot{\theta}_1+R_X^{(2)}=0\\ m\ddot{y}&=R_Y^{(1)} \quad I_e\ddot{\theta}_1+I_p\Omega\dot{\theta}_2-R_Y^{(2)}=0\end{aligned} \tag{6}$$

where $R_Y^{(1)}=\tilde{F}_Y(x_1,y_1)+\tilde{F}_Y(x_2,y_2)$; $R_X^{(1)}=\tilde{F}_X(x_2,y_2)+\tilde{F}_X(x_1,y_1)$;
$R_Y^{(2)}=l_1\tilde{F}_Y(x_1,y_1)-l_2\tilde{F}_Y(x_2,y_2)$; $R_X^{(2)}=l_1\tilde{F}_X(x_1,y_1)-l_2\tilde{F}_X(x_2,y_2)$.

The forces of carrier fluid film of short journal bearing are obtained as:

$$F_x=-\int_0^{L_b}\int_0^{\pi}(\cos(\theta+\phi)p(\theta,z)R)d\theta dz;\quad F_y=-\int_0^{L_b}\int_0^{\pi}(\sin(\theta+\phi)p(\theta,z)R)d\theta dz \tag{7}$$

where L_B is a length of short journal bearing; ϕ is an angle of center lines. It is assumed, that the fluid film is disposed in the region $\theta\in[0;\pi]$. The pressure of the fluid film $p(z_1,\theta)$ is determined from the solution of Reynolds' equation [2]. This solution for the short journal bearing has the following form:

$$p=\frac{3\mu}{h^3}\left[\Omega\frac{\partial h}{\partial\theta}+2\frac{\partial h}{\partial t}\right]z_1(z_1-L_b) \tag{8}$$

where μ is a fluid viscosity; z_1 is local longitudinal coordinate of a journal bearing. The value h is determined as: $h=c+e\cos\theta=c-x\cos(\theta+\phi)-y\sin(\theta+\phi)$, where c is the nominal clearance between the shaft and the bearing.

Future analysis of fluid film forces will be carried out for the journal bearing *A*. The obtained results are true for the journal bearing B, if the general coordinates (x_1,y_1) are changed on (x_2,y_2).

The following dimensionless variables and parameters are used in the future analysis:

$$\tilde{x}_j=\frac{x_j}{c};\quad \tilde{y}_j=\frac{y_j}{c};\quad H=\frac{h}{c};\quad \tau=\Omega t \tag{9}$$

Then the forces of the fluid film can be presented as:

$$\begin{aligned}F_X&=\frac{L_B^3\mu R\Omega}{2c^2}\int_0^{\pi}H^{-3}\cos(\theta+\phi)\{\tilde{x}_1\sin(\theta+\phi)-\tilde{y}_1\cos(\theta+\phi)-2\tilde{x}_1'\cos(\theta+\phi)-2\tilde{y}_1'\sin(\theta+\phi)\}d\theta\\ F_Y&=\frac{L_B^3\mu R\Omega}{2c^2}\int_0^{\pi}H^{-3}\sin(\theta+\phi)\{\tilde{x}_1\sin(\theta+\phi)-\tilde{y}_1\cos(\theta+\phi)-2\tilde{x}_1'\cos(\theta+\phi)-2\tilde{y}_1'\sin(\theta+\phi)\}d\theta\end{aligned} \tag{10}$$

where $H = 1 - \tilde{x}_1 \cos(\theta + \phi) - \tilde{y}_1 \sin(\theta + \phi)$; $(\ )' = \dfrac{d(\ )}{d\tau}$.

The equilibrium position of the rotor under the action of gravity is determined. Then equilibrium of the journal A can be presented as:

$$\tilde{x}_{1,0} = -\varepsilon_1 \cos\phi_{e1}; \quad \tilde{y}_{1,0} = -\varepsilon_1 \sin\phi_{e1}; \quad tg\phi_{e1} = \pi\sqrt{1-\varepsilon_1^{\ 2}} \Big/ (4\varepsilon_1) \tag{11}$$

The dynamics of the rotor with respect to the equilibriums positions is analyzed. Then the change of the variables (5) is rewritten with respect to dimensionless variables: $\tilde{x}_i \to \tilde{x}_i + \tilde{x}_{i,0}$; $\tilde{y}_i \to \tilde{y}_i + \tilde{y}_{i,0}$; $i = 1,2$. Then the nonlinear forces (10) are presented as power series with respect to $\tilde{x}_1, \tilde{y}_1, \tilde{x}_1', \tilde{y}_1'$:

$$\begin{aligned} F_X &= F_{X,0} + F_{X,1}(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1', \tilde{y}_1') + F_{X,2}(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1', \tilde{y}_1') + F_{X,3}(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1', \tilde{y}_1') + \ldots \\ F_Y &= F_{Y,0} + F_{Y,1}(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1', \tilde{y}_1') + F_{Y,2}(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1', \tilde{y}_1') + F_{Y,3}(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1', \tilde{y}_1') + \ldots \end{aligned} \tag{12}$$

where $F_{X,0}; F_{Y,0}$ are constant parts of fluid film forces; $F_{X,1}; F_{Y,1}$ are linear parts of forces with respect to $\tilde{x}_1, \tilde{y}_1, \tilde{x}_1', \tilde{y}_1'$; $F_{X,2}; F_{Y,2}; F_{X,3}; F_{Y,3}$ are nonlinear parts of the forces of the second and the third orders with respect to the general coordinates and velocities. The nonlinear forces (12) are substituted into (6). As the result, the equations of rotor motions have the following matrix form:

$$[M]\ddot{q} + [G]\dot{q} = [K_1]q + [D_1]q' + W(q_1, q_1') \tag{13}$$

where $q = [x, \theta_1, y, \theta_2]^T$; $q_1 = [\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2]^T$; $W(q_1, q_1')$ are vector of nonlinear parts of forces (12). Nonlinear terms within the cubic summands of q and $\dot{q}$ are included in the model of self-sustained vibrations of rotor.

2. THE METHOD OF DYNAMICS ANALYSIS

Now the nonlinear modes for self-sustained vibrations analysis are considered. The motions of the system (13) close to the Hopf bifurcation are analyzed. Then the linear part of the system (13) can be presented as:

$$\dot{z} = \begin{bmatrix} 0 & E \\ -Q & -F \end{bmatrix} z = [\Gamma]\, z \tag{14}$$

where $z = [z_1, \ldots, z_8] = [q\ \dot{q}]^T = [q\ v]^T$; E is an identity matrix. As follows from the results of the numerical simulations, all eigenvalues of the matrix $[\Gamma]$ are complex conjugate; the solution of the system (14) has the following form:

$$z(t) = \sum_{j=1}^{4} \left[C_{2j} W_{2j} \exp(\lambda_{2j} t) + C_{2j-1} W_{2j-1} \exp(\lambda_{2j-1} t) \right] \tag{15}$$

where $\lambda_{2j} = \overline{\lambda}_{2j-1}$; $W_{2j} = \overline{W}_{2j-1}$; $C_{2j} = \overline{C}_{2j-1}$; $\overline{(\ )}$ is denoted the complex conjugation.

If the equilibrium position loses stability, then in this bifurcation point two characteristics exponents have the following values: $\lambda_{1,2} = \pm i\ \chi_1$. The rotor loss of stability describes by the following particular solution of the system (15): $z(t) = C_2 W_2 \exp(\lambda_2 t) + C_1 W_1 \exp(\lambda_1 t)$, where $W_1 = \gamma_1 - i\delta_1$; $C_1 = K_1^{(1)} - iK_1^{(2)}$; $\lambda_1 = \alpha_1 - i\chi_1$; $K_1^{(1)}, K_1^{(2)}$ are constants of integration; $\gamma_1 = \{\gamma_1^{(1)}, \ldots, \gamma_1^{(8)}\}$; $\delta_1 = \{\delta_1^{(1)}, \ldots, \delta_1^{(8)}\}$. This solution can be presented as

$$z_\nu(t) = \gamma_1^{(\nu)} \eta_1(t) + \delta_1^{(\nu)} \eta_2(t); \quad \nu = \overline{1,8} \tag{16}$$

where $$\eta_1(t) = 2\exp(\alpha_1 t)\left[K_1^{(1)}\cos(\chi_1 t) - K_1^{(2)}\sin(\chi_1 t)\right];$$
$$\eta_2(t) = -2\exp(\alpha_1 t)\left[K_1^{(1)}\sin(\chi_1 t) + K_1^{(2)}\cos(\chi_1 t)\right].$$

The following equation is true:

$$x = \gamma_1^{(1)}\eta_1(t) + \delta_1^{(1)}\eta_2(t); \qquad \dot{x} = \gamma_1^{(5)}\eta_1(t) + \delta_1^{(5)}\eta_2(t)$$

These two equations can be rewritten as

$$\eta_1(t) = \frac{x\delta_1^{(5)} - \dot{x}\delta_1^{(1)}}{\gamma_1^{(1)}\delta_1^{(5)} - \gamma_1^{(5)}\delta_1^{(1)}}; \qquad \eta_2(t) = \frac{\dot{x}\gamma_1^{(1)} - x\gamma_1^{(5)}}{\gamma_1^{(1)}\delta_1^{(5)} - \gamma_1^{(5)}\delta_1^{(1)}} \tag{17}$$

Combining (16) and (17), the linear part of the nonlinear mode is obtained as

$$\begin{bmatrix} \theta_1 \\ y \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{y} \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{61} & a_{52} \\ a_{71} & a_{62} \\ a_{81} & a_{72} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{18}$$

where $a_{\nu 1} = \dfrac{\gamma_1^{(\nu)}\delta_1^{(5)} - \delta_1^{(\nu)}\gamma_1^{(5)}}{\gamma_1^{(1)}\delta_1^{(5)} - \gamma_1^{(5)}\delta_1^{(1)}}$; $a_{\nu 2} = \dfrac{\delta_1^{(\nu)}\gamma_1^{(1)} - \gamma_1^{(\nu)}\delta_1^{(1)}}{\gamma_1^{(1)}\delta_1^{(5)} - \gamma_1^{(5)}\delta_1^{(1)}}$; $\nu = 2,...,8;\ \nu \neq 5$.

The nonlinear terms are added into the equation (18) to study nonlinear modes of the self-sustained vibrations. Then the nonlinear mode can be presented as

$$\begin{aligned} q_j &= Q_j(x,v) = a_{j1}x + a_{j2}v + a_{j3}x^2 + a_{j4}v^2 + a_{j5}xv + ... \\ \dot{q}_j &= Q_{j+4}(x,v) = a_{4+j,1}x + a_{4+j,2}v + a_{4+j,3}x^2 + a_{4+j,4}v^2 + a_{4+j,5}xv + ...;\ j = \overline{2,4} \end{aligned} \tag{19}$$

In order to obtain coefficients of the nonlinear part (19) classical Shaw-Pierre nonlinear modes are used [11].

In order to obtain the motions, which are not nonlinear modes, harmonic balance method is used. Then the motions can be presented in the following form:

$$\begin{aligned} &x = A_0 + A_1\cos(\omega t) \qquad \theta_1 = B_0 + B_1\cos(\omega t) + B_2\sin(\omega t) \\ &y = C_0 + C_1\cos(\omega t) + C_2\sin(\omega t) \qquad \theta_2 = D_0 + D_1\cos(\omega t) + D_2\sin(\omega t) \end{aligned} \tag{20}$$

The solutions (20) are substituted in (13); the system of nonlinear algebraic equations with respect to amplitudes and frequency $A_0, A_1, ..., D_2, \omega$ is derived. This system is solved numerically in order to obtain the frequency response.

3. NUMERICAL ANALYSIS

Numerical solution of the nonlinear algebraic system is used to study amplitudes of self-sustained vibrations. Fig. 2 shows the frequency response of the rotor. The eigenvalues of linear system is calculated to obtain the point of the Hopf bifurcation. At $\Omega = 1710\ rad/s$ the equilibrium loses stability and Hopf bifurcation take place. As the result, the unstable self vibrations occur. These unstable limit cycles undergo saddle-node bifurcation at the point A_1. The alternative branch of self vibration was found. This branch marked as $(C_2A_2B_2)$. The curve (C_2A_2) of this branch describes the stable limit cycles, which become unstable at point A_2, where saddle-node bifurcation occur.

Direct numerical integration take place to verify the semi-analytical solutions. The Runge-Kutta method is used. Initial conditions for the direct numerical integration were chosen from the results of harmonic balance method. The calculation results are shown on Fig.2 as points. The solutions obtained by harmonic balance method are in good agreement with numerical simulation of the system.

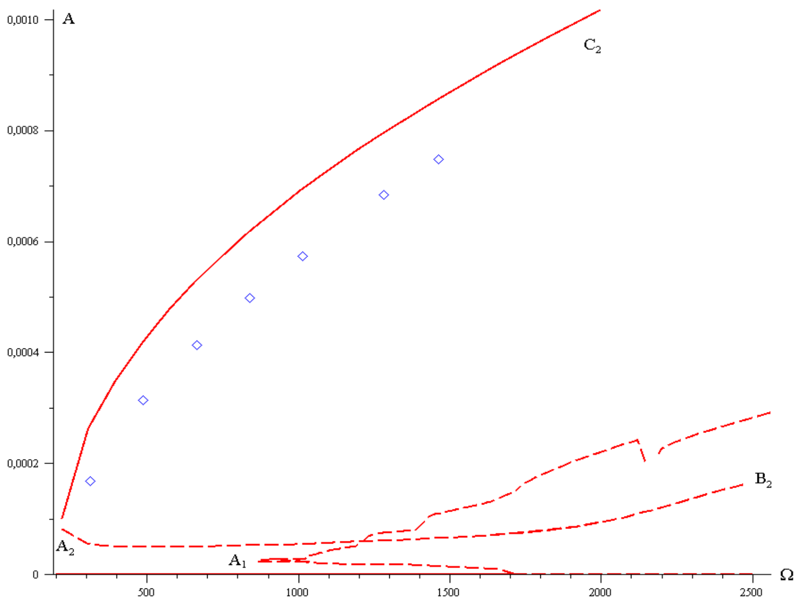

Fig. 2 The frequency response

CONCLUSIONS

Bistability of self vibrations of one disk rotor is investigated. Two types of stable motions are observed at $\Omega \in [300; 1700]\, rad/s$. The first type reflects the uniform rotation of rotor. Self-sustained vibrations occur due to saddle-node bifurcation A_2.

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