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COUPLED DRY FRICTION MODEL FOR THE HEAVY DISK SLIDED WITH SPINNING ON THE ROUGH PLANE.

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ABSTRACT

It is presented a dynamically coupled dry friction model describing the sliding of the heavy rotating disk along the rough plane. The procedure of the models constructing is based on the well known results from the theory of elasticity that tangent stresses lead to shift in the symmetric diagram of the normal contact stresses in the direction of the instantaneous sliding velocity. To use the theory of elasticity results in the dynamics problems, a simple linear approximation of symmetric distribution of normal contact stresses is proposed. The subsequent integration on the spot contact of the net vector and torque differentials yields the exact couple integral model of the sliding and spinning friction. To escape the double integrals calculation in the motion equations, the exact integral expressions are replaced by appropriate Pade expansions. It is shown that the distortion of the symmetry in the distribution of normal contact stresses in the case of circular contact sites results in the appearance of the friction force component directed along the normal to the trajectory of the mass center of the rubbed solids and, consequently, the disk mass center trajectory is declined from the straight line..

INTRODUCTION

The sliding of the heavy rotating disk along the incline plane in the presence of dry friction is one of the classical models in theoretical mechanics. It has been thoroughly studied by numerous authors. In a majority of publications, the authors have used the Coulomb dry friction model, where the force at the point of contact is assumed to be directed opposite to the relative sliding velocity and be independent of its modulus. But there are numerous experimental data testifying that these assumptions do not agree with the real situation in which the interacting bodies simultaneously participate in translation and rotation.

One of the first models describing the relation between the sliding friction and the whirling friction in the case of non-point contact between the moving bodies was proposed by in [1]. A principally new development of the theory was given by in [2], where exact analytic expressions for the resultant vector and the frictional moment for circular contact sites were obtained under the assumption that the distribution of contact stresses in the contact spot obeys the Hertz law. In [2], to apply the obtained dependencies to problems of dynamics, the linear-fractional Pade approximations of these dependencies were constructed. The developed in [2] theory was used in [3] to study the dynamics of a homogeneous circular disk sliding with rotation on a plane. Under the assumption that the distribution of contact stresses obeys the Galin law, exact analytic expressions for the resultant vector and the frictional moment were obtained and their linear-fractional Pade´ approximations were constructed.

The convenience in the use of the Pade´ approximations, which permit describing the effects of combined dry frictions for the entire range of angular and linear velocities, allowed one to construct principally new the two-dimensional coupled models of the sliding and whirling friction the basis of these approximations [4]. All these models were constructed in the assumption that, in the case of circular contact sites, the distributions of normal contact stresses depend only on the position vector with origin at the contact spot center. But, it is known [5] that

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in the case of the rigid solids sliding it is appears tangent stresses that leads to shifting in the symmetric diagram of the normal contact stresses in the direction of the instantaneous sliding velocity. Proposed below models permit to use the theory of elasticity results in the dynamics problems.

1. BASIC RELATIONSHIPS

All described in introduction models were constructed in the assumption that, in the case of circular contact sites, the distributions of normal contact stresses depend only on the position vector with origin at the contact spot center. But, it is known [5] that in the case of the rigid solids sliding it is appears a tangent stress that leads to shifting in the symmetric diagram of the normal contact stresses in the direction of the instantaneous sliding velocity v (Fig. 1).

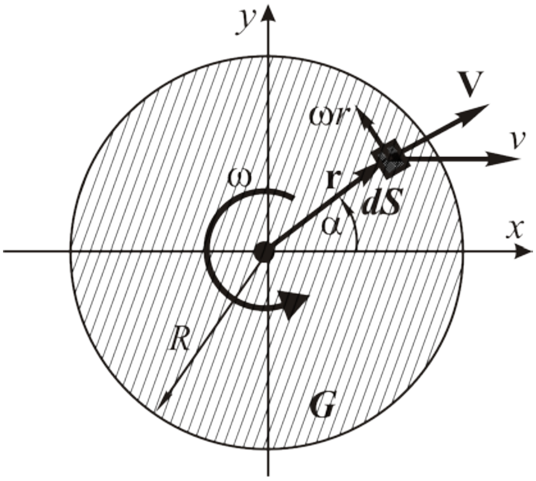

Fig. 1 Kinematics inside the contact spot letters

To use the theory of elasticity results in the dynamics problems, a simple linear approximation of the normal contact stresses distribution is proposed:

$$\sigma(x,y)=\sigma_0(x,y)\left(1+kx/R\right). \tag{1}$$

Typical behavior of the function (1) (red line) is presented on the Fig. 2 in the supposition that symmetric distribution $\sigma_0(x,y)$ (blue line) of the normal contact stresses in the absence of sliding is describing by Galin law:

$$\sigma_0(x,y)=N\left(2\pi R^2\sqrt{1-x^2/R^2-y^2/R^2}\right)^{-1} \tag{2}$$

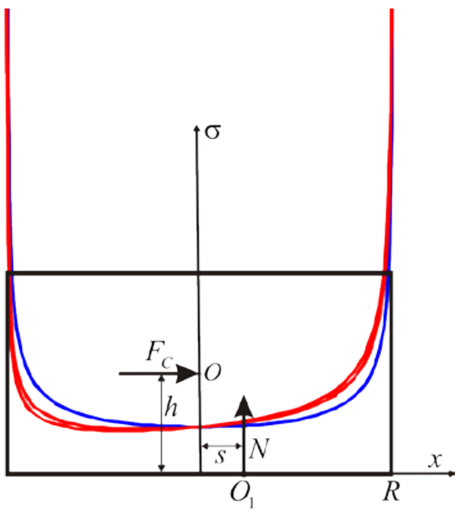

Fig. 2 Distribution of the normal contact stresses

where N - normal reaction, R - disk radius.

To calculate coefficient k in the formula (1) it is used the condition of equality of the external force F torque to the normal reaction force N torque which is appears from the shifting of the center of gravity of the contact spot in the direction of sliding on the value s (Fig.2):

$$Fh = Ns, \ \ N = mg \tag{3}$$

where m - mass of disk and h - distance from disk center mass to the plane of sliding (Fig. 2). On the other hand the shifting s of the disk gravity center relatively of the contact spot center can be defined by the following formula:

$$s = \iint_G x\sigma(x,y)dxdy \Big/ \iint_G \sigma(x,y)dxdy \tag{4}$$

Substitution of the functions (1) and (2) to the (4) yelds: $s = kR/3$. Equalization values s calculated from the formulas (3) and (4) gives

$$k = 3hF/(NR) \tag{5}$$

2. COUPLED MODELS OF THE SLIDING AND SPINNING FRICTION

The combined model of sliding and rolling friction is constructed for circular contact sites under the assumption that the Coulomb law in differential form holds for the small surface element dS in the interior of the contact spot, according to which the differentials of the resultant vector $d\mathbf{F}$ and the moment of friction dM_C with respect to the disk center are determined by the formulas:

$$\begin{gathered} d\mathbf{F} = -f\sigma\frac{\mathbf{V}}{|\mathbf{V}|}\left(1+\mu_1|\mathbf{V}|^3-\mu_2|\mathbf{V}|\right)dS, \ dM_C = -f\sigma\frac{\mathbf{r}\times\mathbf{V}}{|\mathbf{V}|}\left(1+\mu_1|\mathbf{V}|^3-\mu_2|\mathbf{V}|\right)dS, \\ \mathbf{V} = (v-\omega y, \omega x), \mathbf{r} = (x,y) \end{gathered} \tag{6}$$

where f is the coefficient of friction, $\mathbf{r} = (x,y)$ is the position vector of an elemental area in the interior of the contact spot with respect to its center (Fig. 1), ω is the angular velocity of rotation of the contact spot center, but μ_1 and μ_2 are the coefficients which can be defined in practice from experiments.

To obtain the resultant vector and the moment of friction, it is necessary to integrate the expressions (6) over the contact spot. The obtained dependencies, where $F_{\|}$ and $F_{\perp}$ denote the respective components of the resultant vector directed along the tangent and the normal to the trajectory of motion, present an exact combined integral model of sliding and spinning friction

$$\begin{aligned} F_{\|}(u,v) &= -f\iint_G \frac{(v-\omega y)\sigma_0(x,y)}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}}dxdy - \\ &\quad -f\left((\mu_1 v^3-\mu_2 v)\iint_G \sigma_0(x,y)dxdy + 2\mu_1 v\omega^2\iint_G (x^2+y^2)\sigma_0(x,y)dxdy \right) \\ F_{\perp}(u,v) &= -\frac{kf}{R}\iint_G \frac{\omega x^2\sigma_0(x,y)}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}}dxdy \\ M_C(u,v) &= -f\iint_G \frac{(\omega(x^2+y^2)-vy)\sigma_0(x,y)}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}}dxdy - \\ &\quad -f\left((2\mu_1 v^2-\mu_2)\omega\iint_G (x^2+y^2)\sigma_0(x,y)dxdy + \mu_1\omega^3\iint_G (x^2+y^2)^2\sigma_0(x,y)dxdy \right) \\ &u \equiv \omega R, \ G = \{(x,y): x^2+y^2 \le R^2\} \end{aligned} \tag{7}$$

After introducing dimensionless variables: $x = \hat{x}R, y = \hat{y}R$ and $\sigma(\hat{x},\hat{y}) = \hat{\sigma}(\hat{x},\hat{y})N/R^2$ and under the assumption that the distribution of normal contact stresses without spinning has the central symmetry $\sigma_0(x,y) = \sigma(r)$, it is convenient to calculate the modulus of integrals (8) in the polar coordinates: $x = r\cos\varphi,\ y = r\sin\varphi,\ r \in [0,1],\ \varphi \in [0,2\pi]$ (Fig. 1) in which the functions (7) take the form

$$
\begin{aligned}
F_{\|} &= fN\int_0^{2\pi}\int_0^1 \frac{(v - ur\sin\varphi)r\sigma(r)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\varphi}}drd\varphi + 2\pi f\left((\mu_1 v^3 - \mu_2 v)\int_0^R r\sigma(r)dr + 2\mu_1 v\omega^2\int_0^R r^3\sigma(r)dr\right)\\
F_{\perp} &= kfN\int_0^{2\pi}\int_0^1 \frac{ur^3\sigma(r)\cos^2\varphi}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\varphi}}drd\varphi \qquad (8)\\
M_C &= fRN\int_0^{2\pi}\int_0^1 \frac{(ur^2 - vr\sin\varphi)r\sigma(r)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin(\varphi)}}drd\varphi + 2\pi f\left((2\mu_1 v^2 - \mu_2)\omega\int_0^R r^3\sigma(r)dr + \mu_1\omega^3\int_0^R r^5\sigma(r)dr\right)
\end{aligned}
$$

where the "hat" symbol is omitted for brevity, but $\sigma(r) = 1/(2\pi\sqrt{1-r^2})$.

If $k = 0$, then model (8) fully is agree to the model, investigated in [3] and can be considered as the first approximation, but presented in this investigation as the second approximation. Thus, we have substantial approximation to the real situation in dependence on the general properties of the normal contact stresses distribution. The coefficient k in formula (1), (5), (8) is defined by the friction force component $F_{\|}$ from the first expressions in the relations (7-8) and, consequently, the coupled integral friction model which is defined the dynamics of heavy disk on the rough plane under conditions of combined kinematics is

$$
\begin{aligned}
F_{\|} &= fN\int_0^{2\pi}\int_0^1 \frac{(v - ur\sin\varphi)r\sigma(r)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\varphi}}drd\varphi + 2\pi f\left((\mu_1 v^3 - \mu_2 v)\int_0^R r\sigma(r)dr + 2\mu_1 v\omega^2\int_0^R r^3\sigma(r)dr\right)\\
F_{\perp} &= kfN\int_0^{2\pi}\int_0^1 \frac{ur^3\sigma(r)\cos^2\varphi}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\varphi}}drd\varphi,\ k = \frac{3hf}{R}\int_0^{2\pi}\int_0^1 \frac{(v - ur\sin\varphi)r\sigma(r)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\varphi}}drd\varphi \qquad (9)\\
M_C &= fRN\int_0^{2\pi}\int_0^1 \frac{(ur^2 - vr\sin\varphi)r\sigma(r)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin(\varphi)}}drd\varphi + 2\pi f\left((2\mu_1 v^2 - \mu_2)\omega\int_0^R r^3\sigma(r)dr + \mu_1\omega^3\int_0^R r^5\sigma(r)dr\right)
\end{aligned}
$$

Plot of tangent friction force component $F_{\|}/F_0$ as function of velocity of sliding v at the constant velocity of whirling $u = 1$ (left figure) and plot of the normal friction force component $F_{\perp}/(\mu F_0)$ as function of $k = v/u$ (right figure) are presented on the fig. 3.: As concerned friction torque then, qualitatively, its behavior is the same as case of using classical form Coulomb law: there are only small quantitative distinctions.

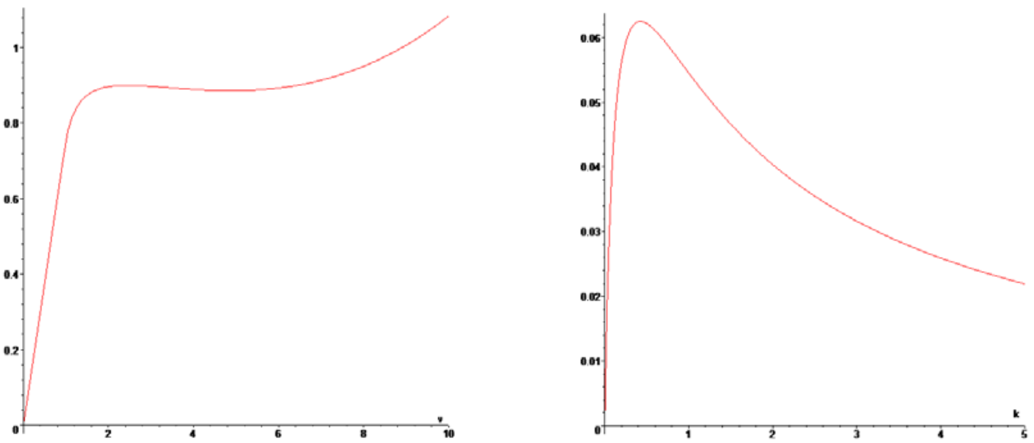

Fig. 2 Tangent and normal friction force components

The expressions for the components of the resultant vector and the moment of friction in relations (9) have several important properties as functions of u and v.

Property 1. The distortion in symmetric diagram of the normal contact stresses distribution results in the appearance of the resultant vector component $F_\perp$ directed along the normal to the trajectory of motion. The resultant vector is not directed opposite to the velocity of sliding.

Property 2. The distortion in the symmetric diagram of distribution of normal stresses does not affect to the moment M_C and the resultant vector component $F_\|$ directed along the tangent to the trajectory.

Property 3. The tangent $F_\|$ and normal $F_\perp$ components of the friction force, just as the moment M_C, are homogeneous functions of the variables u and v of zero order of homogeneity and hence are invariant under the similarity group:

Property 4. The expressions (9), for the moment and both components of the friction force as functions of u and v have a singularity at the point $(u,v)=(0,0)$, because they do not have any limit at this point with respect to both of the variables u and v.

Property 5. In the case of pure sliding $u=0$ or spinning $v=0$, the moment M_C and the tangential component $F_\|$ are homogeneous models corresponding to the usual Coulomb law:

$$F_\|(0,v)=F_0\equiv fN,\ M_C(u,0)=M_0, M_0=\pi fNR/4$$

Property 6. In the case of pure sliding, the normal component vanishes: $F_\perp(0,v)=0$, and hence the friction force is directed opposite to the velocity vector; in the case of pure spinning, it is equal to $F_\perp(u,0)=\mu F_0,\ \mu=3\pi hf/(4R)$.

Property 7. The moment M_C and both components of the friction force $F_\|$ and $F_\perp$ have only one nonzero first partial derivative (the others are zero):

$$\left.\frac{\partial M_C}{\partial u}\right|_{u=0}=\frac{M_0}{3v},\ \left.\frac{\partial F_\|}{\partial v}\right|_{v=0}=\frac{\pi F_0}{4u},\ \left.\frac{\partial F_\perp}{\partial u}\right|_{u=0}=\frac{4\mu F_0}{9\pi v}$$

The integral models (9) give a good description of the combined sliding and spinning friction, but are inconvenient to be used in problems of dynamics, because it is required to calculate multiple integrals in the right-hand sides of the equations of motion. This difficult procedure can be eliminated by replacing the exact integral expressions by the corresponding Pade approximations. The simplest of them is the linear-fractional approximation preserving the value at zero and at infinity of both for the torque M_c and for the tangent force component $F_\|$. But, for the normal friction force component, corresponded Pade approximation, naturally, became of the second order.

$$\begin{aligned}
&M_C=M_0\left(\frac{u}{u+mv}+2\pi\left((2\mu_1v^2-\mu_2)uI_3+\mu_1u^3I_5\right)\right),\ \frac{1}{m}=\frac{v}{M_0}\left.\frac{\partial M_C}{\partial u}\right|_{u=0}\equiv\frac{1}{3}\\
&F_\|=F_0\left(\frac{v}{v+au}+2\pi\left((\mu_1v^3-\mu_2v)I_1+2\mu_1vu^2I_3\right)\right),\ \frac{1}{a}=\frac{u}{F_0}\left.\frac{\partial F_\|}{\partial v}\right|_{v=0}\equiv\frac{\pi}{4}\\
&F_\perp=\frac{\mu F_0uv}{(u+bv)(v+au)},\ \frac{1}{b}=\frac{v}{\mu F_0}\left.\frac{\partial F_\perp}{\partial u}\right|_{u=0}\equiv\frac{4}{9\pi}
\end{aligned}\tag{10}$$

The linear-fractional Pade´ approximations (10) preserve the values of the functions $F_\|(u,v)$ and $M_C(u,v)$ at zero, as well as their behavior and the behavior of their first derivatives at infinity. But model of this type cannot completely preserve the values of all first partial derivatives of these functions at zero. To obtain a correct description of the behavior of the first derivatives at zero, it is required to use the second-order Pade´ approximations, and then the coupled model of sliding and spinning friction takes the form

$$M_C = M_0\left(\frac{u^2+muv}{v^2+muv+u^2}+2\pi\left((2\mu_1 v^2-\mu_2)uI_3+\mu_1 u^3 I_5\right)\right),\ m=\frac{v}{M_0}\frac{\partial M_C}{\partial u}\bigg|_{u=0}\equiv\frac{1}{3}$$

$$F_{\parallel} = F_0\left(\frac{v^2+auv}{v^2+auv+u^2}+2\pi\left((\mu_1 v^3-\mu_2 v)I_1+2\mu_1 vu^2 I_3\right)\right),\ a=\frac{u}{F_0}\frac{\partial F_{\parallel}}{\partial v}\bigg|_{v=0}\equiv\frac{\pi}{4} \tag{11}$$

$$F_{\perp} = \frac{\mu F_0 uv}{(u+bu)(v+u/a)},\ \frac{1}{b}=\frac{v}{\mu F_0}\frac{\partial F_{\perp}}{\partial u}\bigg|_{u=0}\equiv\frac{4}{9\pi}$$

The second-order model (11) completely satisfies all properties 1–7 of the exact integral models (9). But, for the majority of the problems of dynamics, it is sufficient to use the order model (10). The second-order model (11) is required for a more precise qualitative analysis, for example, for determining the boundaries of the stagnant region and the motion stopping time.

The approximations (10) and (11) hold for positive values of u and v. They can be easily generalized to the case of arbitrary (in sign) velocities u and v by a formal change by absolute values in the denominators of the corresponding expressions.

The use of the friction models based on the Pade´ expansions allows one to avoid calculations of multiple integrals over the contact spot, which significantly simplifies their use in problems of dynamics. Moreover, the models (10) and (11) can be considered as the phenomelogical models. To obtain a correct description of the combined sliding and spinning dry friction in the complete statement based on the models (10) and (11), it is necessary to know at most six coefficients, which can be determined experimentally in solving the real practical problems.

CONCLUSIONS

It is developed a dynamically coupled integral dry friction model describing the sliding of the heavy rotating disk along the rough plane. To escape the double integrals calculation in the motion equations, the exact integral expressions are replaced by appropriate Pade expansions.

It is shown that the distortion of the symmetry in the distribution of normal contact stresses in the case of circular contact sites results in the appearance of the friction force component directed along the normal to the trajectory of the mass center of the rubbed solids and, consequently, the disk mass center trajectory is declined from the straight line.

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