CRITICAL RE-EXAMINATION OF ADOMIAN'S DECOMPOSITION AND HOMOTOPY PERTURBATION METHODS IN NONLINEAR MECHANICS

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ABSTRACT

This paper was provides a systematic account of Adomian's decomposition method and the homotopy perturbation method from the standpoint of the theory of expansion in parameter for an approximate analytical integration of systems of equations in nonlinear dynamics. It was shown that approximations which have got by using those methods represent the expansions of the exact solution in the form of the Taylor series for the independent variable. A modified method of the continuation parameter is proposed, which combines both approaches. The method allows simplifying the calculations, both at the model building phase, and for its further use. Two-dimensional approximation was used of Padé type, which has shown its effectiveness for acceleration of approximations convergence and their analytic continuation. On the basis of the proposed method it was calculated solution for Riccati equation of special type, which is widely used in theory of solitons.

INTRODUCTION

Adomian's decomposition method (ADM) [2,3,5] and homotopy perturbation method (HPM) [8] are widely used for an approximate analytical integration of equations in nonlinear dynamics. Recently, they have a significant impact on development of the theory of analytical solution of nonlinear equations with strong nonlinearity.

Convergence of the ADM is examined by a number of authors in different ways [2,5]. This problem for HPM is studied in Ref. [12] for the case of algebraic equations only. At the same time, the authors of the HPM and their followers do not analyze in the above cited papers the type of homotopy mapping and the properties of the obtained approximations (the existence, the area of applicability, stability, rate of convergence, etc.).

For ODEs with polynomial terms the ADM and HPM give a solution in the form of a polynomial (a series) in powers of the integration variable. Terms in the other type of ODEs can be approximated by Taylor series in powers of independent variable, desired function and its derivatives. This version of ADM is called the modified ADM (MADM) [10].

It is worth mentioning that ADM and HPM can be satisfactorily applied only with an effective method of summation. The most natural analytical continuation method is Padé approximants (PAs) [4,5]. PAs effectively solves the problem of analytical continuation of power series, and this is a basis of their successful application in the study of applied problems. Currently, the method of PAs is one of the most promising non-linear methods of summation of power series, and the localization of its singular points. PAs have turned into quite a separate section in the approximation theory, and they found a variety of applications in the study of differential equations depending on a parameter. Recently, the method of PAs for single-variable functions (1-D PAs) has been successfully extended to the approximation of two variable functions (2-D PAs) [13].

In some cases, solutions of both ADM and HPM methods coincide, but this is not always the case. A natural question arises about the correlation of ADM and HPM. Note that for non-linear

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algebraic equations, this question was recently resolved in Ref. [9] and a coincidence of approximations due to certain selection of parameters was shown. In the case of ODEs, both methods can be combined on the basis of a new synthetic approach, the development of which this work is devoted to.

1. A SYSTEMATIC ACCOUNT AND COMPARISON OF METHODS

Let us introduce a formal definition of ADM and HPM for systems of ODEs using the terminology of the perturbation method. It is known that in vicinity of regular point any ODE or system of ODEs may be represented by a normal system of ODEs of the first order in respect to the unknown functions $\{u_i = u_i(\xi)\}_{i=1}^{n}$ on the interval $\Omega: \xi \in]0,1[$ with the BC on the bounds $\partial\Omega: \xi = 0 \cup 1$

$$Lu_i + R_i(\xi, u_1, ..., u_n) + N_i(\xi, u_1, ..., u_n) = g_i(\xi), G_i(u_1, ..., u_n)\big|_{\partial\Omega} = 0, L = \frac{d}{d\xi}, i = \overline{1,n} \quad (1)$$

Here L and R_i are the linear operators, N_i and G_j are the non-linear operators. We assume also that the point $\xi_0 = 0$ belongs to closure Ω, and R_i, N_i and G_j are the holomorphic functions for $\{u_i\}_{i=1}^{n}$. According to ADM, the solution represents in the form

$$Lu_{ij}^{A} = A_{ij}, u_i = \sum_{j=0}^{\infty} u_{ij}^{A}, i = \overline{1,n}, j = \overline{0,\infty} \quad (2)$$

where A_{ij} are the Adomian's polynomials [3], defined by the formulas

$$A_{i0} = g_i, A_{ij} = -\frac{1}{j!}\frac{\partial^j}{\partial\lambda^j}\left(N_i\left(\sum_{m=0}^{j} u_{im}^{A}\lambda^m\right) + R_i\left(\sum_{m=0}^{j} u_{im}^{A}\lambda^m\right)\right)\Bigg|_{\lambda=0}, i = \overline{1,n}, j = \overline{1,\infty} \quad (3)$$

As it was shown in Ref. [5], ADM is equivalent to the perturbation of the governing equation and its solution in respect of parameter λ which is introduced as follows

$$u_i = \sum_{j=0}^{\infty} u_{ij}^{A}\lambda^j, \ Lu_i + \lambda\left(R_i(u_1, ..., u_n) + N_i(u_1, ..., u_n)\right) = g_i, i = \overline{1,n}$$

According to the HPM governing BVP has to be written in the following form

$$\begin{gathered} Lu_i - Lu_i\big|_{\partial\Omega} + R_i\left(\{u_k\}_{k=1}^{n}\right) + N_i\left(\{u_k\}_{k=1}^{n}\right) + F\left(Lu_1, \{u_k\}_{k=1}^{n}\right)\delta_i^1 - g_i = 0 \\ G_i\left(\left\{u_k\big|_{\partial\Omega}\right\}_{k=1}^{n}\right)\Bigg|_{\partial\Omega} = 0, i = \overline{1,n} \end{gathered} \quad (4)$$

where F is the non-linear differential operator, δ_i^1 is the Kronecker's delta, $u_i\big|_{\partial\Omega}$ are the so-called «trial» functions that satisfy the BCs [8]. We have to introduce a parameter ε as follows to obtain the sequence of BVPs for the HPM

$$Lu_i + \varepsilon\left(Lu_i\big|_{\partial\Omega} + R_i + N_i + F\delta_i^1 - g_i\right) = 0, G_i\left(\left\{u_k\big|_{\partial\Omega}\right\}_{k=1}^{n}\right)\Bigg|_{\partial\Omega} = 0, u_i = \sum_{j=0}^{\infty} u_{ij}^{H}\varepsilon^j, i = \overline{1,n} \quad (5)$$

Consider $\{u_i = u_i(\xi)\}_{i=1}^{n}$ and their derivatives as independent arguments, we introduce operators R_i, N_i, F and G_j as multidimensional Taylor series

$$R_i + N_i = \sum_{j=1}^{n}\left(N_{ij}u_j + \frac{1}{2}\sum_{p=1}^{n} N_{ijp}u_ju_p + \dots\right), F = \left(F_0 Lu_1 + \frac{1}{2!}\sum_{p=1}^{n} F_{0p}u_pLu_1 + \dots\right) + \sum_{j=1}^{n}\left(F_ju_j + \frac{1}{2}\times \right.$$

$$\left. \times \sum_{p=1}^{n} F_{jp}u_ju_p + \dots\right), G_i = \sum_{q=1}^{n}\left(G_{iq}\left(u_q - u_q\big|_{\partial\Omega}\right) + \frac{1}{2}\sum_{p=1}^{n} G_{iqp}\left(u_q - u_q\big|_{\partial\Omega}\right)\left(u_p - u_p\big|_{\partial\Omega}\right) + \dots\right), i = \overline{1,n} \quad (6)$$

$$N_{ij} = \sum_{r=0}^{\infty} N_{ij}^r\xi^r,\ N_{ijp} = \sum_{r=0}^{\infty} N_{ijp}^r\xi^r, \dots, F_i = \sum_{r=0}^{\infty} F_i^r\xi^r,\ F_{ij} = \sum_{r=0}^{\infty} F_{ij}^r\xi^r, \dots\ g_i = \sum_{j=0}^{\infty} g_{ij}\xi^i,\ i, j = \overline{1,n}$$

Let's substitute series (6) into Eqs. (3)

$$A_{ij} = -\frac{1}{j!}\frac{\partial^j}{\partial\lambda^j}\left(\sum_{r=1}^{n}\left(N_{ir}u_r + \frac{1}{2!}\sum_{p=1}^{n} N_{irp}u_ru_p + \dots\right)\right)\Bigg|_{\lambda=0} = -\sum_{r=1}^{n}\left(N_{ir}u_{rj}^A + \sum_{p=1}^{n} N_{irp}\sum_{k=0}^{j} u_{rk}^A u_{p(j-k)}^A + \dots\right) \quad (8)$$

We obtain successive approximations for the ADM:

$$u_i = \xi^0\left(\left[u_i\big|_{\partial\Omega}\right] + [0] + [0] + \dots\right) +$$

$$+\xi^1\left([g_{i0}] + \left[-\left(\sum_{j=1}^{n}\left(N_{ij}^0 + \frac{1}{2!}\sum_{p=1}^{n} N_{ijp}^0\, u_j\big|_{\partial\Omega}\, u_p\big|_{\partial\Omega}\right) + \dots\right)\right] + [0] + \dots\right) + \dots, i = \overline{1,n} \quad (9)$$

For the HPM one obtains:

$$u_1 = \xi^0\left(\left[u_1\big|_{\partial\Omega}\right] + [0] + [0] + \dots\right) + \xi^1\left([0] + \left[g_{10} - \sum_{r=1}^{n}\left(\left(N_{1r}^0 + F_r^0\right)u_r\big|_{\partial\Omega} + \right.\right.\right.$$

$$\left.\left. \frac{1}{2!}\sum_{p=1}^{n}\left(N_{1rp}^0 + F_{rp}^0\right)u_r\big|_{\partial\Omega}\, u_p\big|_{\partial\Omega} + \dots\right)\right] + \left[-\left(F_0^0 + F_{00}^0\sum_{p=1}^{\infty} u_p\big|_{\partial\Omega}\right)\left(g_{10} - \sum_{r=1}^{n}\left(\left(N_{1r}^0 + F_r^0\right)u_r\big|_{\partial\Omega} + \right.\right.\right.$$

$$\left.\left.\left.\left. + \frac{1}{2!}\sum_{p=1}^{n}\left(N_{1rp}^0 + F_{rp}^0\right)u_r\big|_{\partial\Omega}\, u_p\big|_{\partial\Omega} + \dots\right)\right)\right] + \dots\right) + \dots, u_i = \xi^0\left(\left[u_i\big|_{\partial\Omega}\right] + [0] + [0] + \dots\right) + \quad (10)$$

$$+\xi^1\left([0] + \left[g_{i0} - \sum_{r=1}^{n}\left(N_{ir}^0u_r\big|_{\partial\Omega} + \frac{1}{2!}\sum_{p=1}^{n} N_{irp}^0\, u_r\big|_{\partial\Omega}\, u_p\big|_{\partial\Omega} + \dots\right)\right] + [0] + \dots\right) + \dots,\ i = \overline{2,n}$$

The brackets contain the expression corresponding to the successive approximations in powers of the parameter.

Comparison of the ADM and HPM are based on the comparison of the nonlinear operators in ODEs in normal and general forms. If one takes into account terms up to the second order in the Eqs. (6), the normal form of Eqs. (4) can be written as follows

$$Lu_1 + \sum_{j=1}^{n}\left(\frac{N_{1j} + F_j}{1+F_0}u_j + \frac{1}{2}\sum_{p=1}^{n}\frac{\left(N_{1jp} + F_{jp}\right)\left(1+F_0\right) - F_{0p}\left(N_{1j} + F_j\right) - F_{0j}\left(N_{1p} + F_p\right)}{2\left(1+F_0\right)^2}u_ju_p\right) =$$

$$= \frac{g_1}{1+F_0},\ Lu_i + R_i\left(u_1,\dots,u_n\right) + N_i\left(u_1,\dots,u_n\right) = g_i,\ i = \overline{2,n} \quad (11)$$

In other words, in this case ADM applied to Eqs. (11) is equivalent to the HPM applied to Eqs. (4).

There are several special cases that are interesting to consider from a practical point of view. Thus, for singular perturbed nonlinear equations for $F_j \equiv F_{jp} \equiv 0,\ j, p = \overline{1,n}$ from the Eqs. (9) - (10) we obtain

$$Lu_1+\sum_{j=1}^{n}\left(\frac{N_{1j}}{\varepsilon}u_j+\frac{1}{2}\sum_{p=1}^{n}\frac{N_{1jp}}{2\varepsilon}u_ju_p\right)=\frac{g_1}{\varepsilon},u_1=\xi^0\,u_1\big|_{\partial\Omega}+\xi^1\Bigg(\Bigg(g_{10}-\sum_{r=1}^{n}\Bigg(N_{1r}^0\,u_r\big|_{\partial\Omega}+$$

$$+\frac{1}{2!}\sum_{p=1}^{n}N_{1rp}^0\,u_r\big|_{\partial\Omega}u_p\big|_{\partial\Omega}+...\Bigg)-(\varepsilon-1)\Bigg(g_{10}-\sum_{r=1}^{n}\Bigg(N_{1r}^0\,u_r\big|_{\partial\Omega}+\frac{1}{2!}\sum_{p=1}^{n}N_{1rp}^0\,u_r\big|_{\partial\Omega}u_p\big|_{\partial\Omega}+...\Bigg)\Bigg)+...\Bigg)+...$$

In other words, the coefficients in the power of the independent variable in the HPM solution represent expansion of coefficients of ADM on the natural small parameter in the vicinity of $x=1$.

2. USING THE PAS

From Eqs. (10) and (11) we see that, if the equation is solved in respect to the highest derivative, the coefficients with the same degree of variable solutions ADM and HPM converges each to other as far as order of approximation increases. It was shown in Ref. [5] that the solution of the ADM converges to the decomposition of exact solution in Taylor series in the area of its holomorphy in the vicinity of x = 0. That is the reason that the same properties will have a solution of HPM in the case when equation is in normal form. This allows the use meromorphic continuation in the form of PAs [5].

For ADM such a continuation procedure was proposed in Ref. [5]. Later, this approach has been developed by a number of authors [1, 7], and was named the modified Adomian's decomposition method and PAs (MADM-Padé). Thus, it is possible to use PAs to HPM with modifications, by decomposition of nonlinear terms in the series as for the independent variable, so for the desired function (MHPM-Padé).

2-D PAs in the form of V. Vavilov [13] has a great promise for use as an analytic continuation. This technique allows us to choose the coefficients of 2-D Taylor series for construction of an unambiguous 2-D PA with a given structure of the numerator and denominator, as well as ensures optimal PAs features in the sense of Theorem Montessus de Ballore-type. That is means the homogenous convergence of PA to approximated function with increasing of the degree of the numerator and the denominator in all points of its meromorphy area. It should be noted that direct application of 2-D PAs does not leads to the anticipated merging of 1-D approximations. This is due to the initial requirements to the 2-D approximation to ensure its transition to 1-D in the case when the second variable equal to zero [13]. At the same time as for the method of parameter continuation it is necessary to ensure such a transition when parameter is equal to one. This can be achieved by combining of this method with 2-D PAs from a converted parameter, which maps the unit to zero.

3. MODIFIED METHOD OF THE PARAMETER CONTINUATION

The modified method of the parameter continuation (MMPC) proposed in this paper consists of perturbation technique of special form and the analytic continuation of obtained approximations by PAs. It coincides with the HPM for $F\equiv 0$, and with the ADM - when $g\equiv 0$, and thus generalizes them. The method does not imply the introduction of «trial» functions that satisfy the BC, they will be satisfied in successive approximations, and this gives us an opportunity to solve the BVP with complicated BCs [5]. To implement the MMPC, we introduce a parameter ε as follows

$$Lu_i=\varepsilon\left(g_i-R_i\left(\{u_k\}_{k=1}^n\right)-N_i\left(\{u_k\}_{k=1}^n\right)\right),G_i\left(\left\{u_k\big|_{\partial\Omega}\right\}_{k=1}^n\right)\Bigg|_{\partial\Omega}=0,u_i=\sum_{j=0}^{\infty}u_{ij}^M\varepsilon^j,i=\overline{1,n} \qquad (13)$$

Substitute the power series into Eqs. in (13) and split it with respect to the powers of ε, after summation of the coefficients with the same degrees of ξ for $\varepsilon=1$, we get

$$u_i=\xi^0\left(\left[u_i\big|_{\partial\Omega}\right]+[0]+[0]+...\right)+$$
$$+\xi^1\left([0]+\left[g_{i0}-\sum_{r=1}^{n}\left(N_{ir}^0u_r\big|_{\partial\Omega}+\frac{1}{2!}\sum_{p=1}^{n}N_{irp}^0\,u_r\big|_{\partial\Omega}u_p\big|_{\partial\Omega}+...\right)\right]+[0]+...\right)+...,i=\overline{1,n} \qquad (14)$$

Approximation thus obtained is converted to 1-D PA in respect to ξ or 2-D PA.. MMPC approximation is simpler than ADM and HPM. Analysis of the obtained approximation suggests that, in contrast to the ADM and HPM, it gives the exact value of the coefficients in the power of the independent variable to the extent equal to the order of approximation (taking into account the expansion in power series of expressions in the equation). This guarantees the stability of computation with a limit-order approximation of the independent variable.

One of the possible fields for application of the proposed approach is the nonlinear problems of plates and shells dynamics theory [12]. The equations of dynamics of geometrically nonlinear thin-walled structures can be reduced to the resolving equations which contain the products and squares of the desired functions and their derivatives. In this case the solution of Eqs. (14) can be written without three dots.

4. NUMERICAL RESULTS

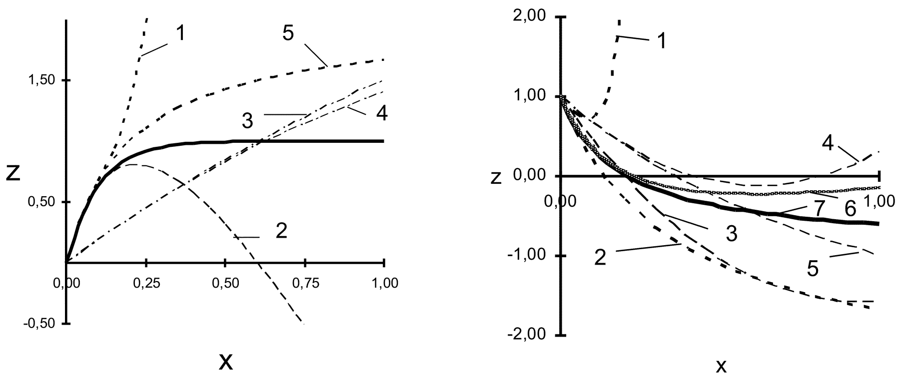
Let us consider the computational aspects of the proposed approach on following BVPs

$$\alpha z' + z = 1, z(0) = 0, x \geq 0, \alpha = 0{,}1 \tag{15}$$

$$\alpha z' = -z^2 + x - 1, \; z(0) = 1, x \geq 0, \alpha = 0{,}2 \tag{16}$$

We consider three types of PAs - on the independent variable $z_{(x)}$, on the specified parameters $z_{(\varepsilon)}$, and 2-D - $z_{(2)}$. Typical behavior of approximations for the BVP (15) shown in Fig. 1a. ADM approximation describes the exact solution well only for a distance which is comparable with the value of the natural small parameter α. Despite the fact that the error of solutions HPM is substantially less than the ADM, HPM is not accurately reflect the nature of solutions, namely the phenomenon of boundary layer in vicinity to zero. At the same time, PAs for the ADM approximations for independent variable and PAs for the MMPC (1-D and 2-D) give satisfactory qualitative and quantitative results.

A more complex picture arises when considering the significantly non-linear inhomogeneous ODE, for example, the type of special Riccati equation (16). For the Riccati equation, which is not solved in quadrature, cites a number of problems of optimal control theory, in some cases nonlinear differential equations of Painlevé are reduced to it, which are successfully used now in the theory of solitons. Fig. 1b shows graphs of approximations for BVP (16).



a) b)

Fig. 1 Solusions of BVPs by different methods.

a – for BVP (15): solid line - the exact solution, 1 – three terms ADM, 2 – $z_{(\varepsilon)}$ for ADM, 3 – three terms HPM, 4 – $z_{(x)}$ for HPM, 5 – 2-D Padé $z_{(2)}$ for MMPC, ADM and HPM; .b - for BVP (16):1 – three terms ADM, 2 – $z_{(\varepsilon)}$ for ADM, 3 – $z_{(\varepsilon)}$ for ADM, , 4 – three terms HPM, 5 – $z_{(x)}$ for HPM, 6 – $z_{(\varepsilon)}$ for HPM and MMPC, 7 – $z_{(x)}$ and 2-D Padé $z_{(2)}$ for MMPC

The graphs show that the solution is described well by HPM approximation and MHPM-Padé «in average», and badly - in the boundary layer. ADM approximation and MADM-Padé, on the contrary, is in good agreement with the behavior of solution in vicinity of zero and in the bad one - on the stationary part. At the same time, 1-D and 2-D PAs, based on approximations of the MMPC, is well described the solution on the whole interval.

CONCLUSIONS

This paper gives a systematic description of Adomian's decomposition method and homotopy perturbation method. Here were obtained analytical expressions for calculating the coefficients of approximation for these methods. We conducted a comparison between the approximations and identified the conditions when they coincide.

We propose a modified method of the parameter continuation (MMPC) that combines both approaches. This method allows to simplifying the calculations both at the stage of constructing the model, and also within its continuation use due to the precise values of the Taylor coefficients for the solution of the degree which is not exceeding the number of approximation. We present the expression to calculate approximations by the MMPC in the general case and with the nonlinearity type of products and squares of the desired functions.

We analyzed using of fractional-rational transformation for the polynomial approximation in the form of the 1-D and 2-D PAs which is used to increase degree of convergence and for analytic continuation of the approximation in the region of its meromorphy. It was concluded that such a transformation is justified if it is applied to polynomials which depend on the variable of integration. We used 2-D PAs for the independent variable and for the artificial parameter using the scheme of V.°Vavilov. In this paper we have shown that this transformation provides a satisfactory quality for the approximation behavior and minimize its error, in spite of the fact that usage of 2-D PAs requires further theoretical justification.

We conducted a study of numerical results by applying the methods for model examples which were perturbed with natural small parameter. It is shown that the application of PAs provides them a sufficient accuracy in the studied area. In this paper it is shown the advantage of the approximations which were obtained on the basis of the MMPC.

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