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EXISTENCE AND STABILITY OF DISCRETE BREATHERS WITH DIFFERENT SYMMETRIES IN 2D SQUARE LATTICES

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ABSTRACT

Symmetry-related invariant manifolds admitting existence of localized vibrations in square lattice are found. Discrete breathers on these manifolds and their stability are analyzed for a case of homogeneous potentials of different degrees.

INTRODUCTION

Discrete breathers (DBs) represent spatially localized and time-periodic excitations in nonlinear Hamiltonian lattices [1, 2]. During two last decades these dynamical objects were studied by different analytical, numerical and experimental methods in a great number of papers. However, the majority of papers treat discrete breathers in one-dimensional chains and only few articles discuss these dynamical objects in 2D and 3D periodic structures [3 – 7].

In the present paper, we consider discrete breathers in 2D square lattice with one degree of freedom per lattice site (we refer to it as scalar model of square lattice). Different physical interpretation can be given to this mathematical model. For example, it have been used for describing transversal mechanical vibrations of the plane lattice in [3], charge vibrations in an electrical network of nonlinear capacitors coupled to each other with linear inductors [5], etc.

DBs in the above mentioned scalar model of square lattice for the case of homogeneous potentials of different degrees were found with high precision in [3]. However, the problem of stability of the obtained breathers was not considered in this paper.

We add on-site potential to the model [3] and treat the problem of the breather stability with respect to the relative strength of on-site and inter-site parts of the potential energy U of the considered system. Moreover, we develop a group-theoretical approach for finding symmetry-related invariant manifolds which can simplify essentially the studying of DBs in 2D and 3D periodic structures independently of the type of the interparticle interaction potential.

1. INVARIANT MANIFOLDS

Different vibrational regimes of any nonlinear physical system can be classified by subgroups of the “parent” symmetry group consisting of all transformations which do not change the system dynamical equations. This idea was used for constructing *bushes* of *extended* nonlinear normal modes in physical systems with discrete symmetry (see [8, 9, 10]). Obviously, the same idea can also be used for classification of *localized* nonlinear modes in 2D and 3D periodic structures.

We discuss existence and stability of DBs in two-dimensional square lattice whose equilibrium state symmetry is described by space (plane) group $G_0=C_{4v}^1$.

We consider a scalar dynamical model associated with this lattice admitting that only one dynamical variable $q_{ij}(t)$ corresponds to (i,j) site of the lattice (i=1..*N*, j=1..*M*). In general case, dynamical equations of our model can be written as follows

$$\ddot{Q}_{NxM} = F(Q_{NxM}) \quad (1)$$

Here matrix

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$$Q_{NxM} = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1N} \\ q_{21} & q_{22} & \cdots & q_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ q_{M1} & q_{M2} & \cdots & q_{MN} \end{pmatrix} \quad (2)$$

determines the set of all variables $q_{ij}(t)$ corresponding to the $N \times M$ fragment of the considered lattice.

It can be shown that dynamical equations (1) are invariant under all transformations generated by symmetry elements of the parent group C_{4v}^1. Moreover, for an even potential energy U, there is an additional symmetry transformation P which changes the signs of all the particles without their transposition. Since DBs represent localized dynamical objects, they can be classified by *point* subgroups of the parent group $G_0 = C_{4v}^1 \times (E, P)$, where E is identical element. To find such classification, we first look for symmetry-related *invariant manifolds* of the dynamical equations with the aid of the methods which were previously developed in the framework of the theory of phase transitions in crystals [11].

We will investigate *strongly localized* discrete breathers and, therefore, it is sufficient to deal with *small* lattice fragments (2). We choose concrete values of N and M from the condition that amplitudes of $q_{ij}(t)$ for peripheral sites must be much smaller than those for the breather core.

If a given fragment $Q_{NxM}(t)$, possesses the symmetry group $G \subset G_0$, then it is *invariant* under the action of all symmetry elements $g_i \in G$:

$$\hat{g}_i Q_{NxM}(t) = Q_{NxM}(t), \quad \forall g_i \in G .$$

Here $\hat{g}_i$ is operator acting in $N \times M$ functional space which is induced by the symmetry element g_i of the group G. It acts on an arbitrary function $f(\vec{r})$ in line with the conventional definition [12]:

$$\hat{g}_i f(\vec{r}) = f(g_i^{-1} \vec{r}).$$

In practice, as a rule, the subgroups G_i of the parent group G_0 ($G_i \subset G_0$) are unknown, as well as the invariant manifolds corresponding to them.

In [11], we described an algorithm which allows one to single out invariant subspaces corresponding to all possible subgroups of a given parent group. To this end, we first find invariant subspaces of all the individual matrices of the natural representation of the group G_0. Then we find subsequently all possible *intersections* of these subspaces, because each *intersection* corresponds to a subgroup which is a *union* of those subgroups who correspond to these subspaces.

This way (see [11] for details) provides us with all the *nonequivalent* subspaces of the configuration space simultaneously with the complete list of corresponding subgroups of the given parent group G_0.

Each of the above invariant subspaces represents an invariant manifold relative to time evolution of our dynamical model described by Eqs. (1). However, not all of these manifolds can be used for constructing discrete breathers, since the *structure* of some of them does not permit existence of *localized* vibrations. Indeed, the manifold

$$\begin{pmatrix} c & b & c \\ b & a & b \\ c & b & c \end{pmatrix}$$

allows localized mode when $|a| >> |b| >> |c|$, while the manifold

$$\begin{pmatrix} b & a & c \\ b & a & c \\ b & a & c \end{pmatrix}$$

does not admit localization, because variables q_{ij}, expressed via *a, b, c* have no tendency to decrease by amplitude from its center to periphery.

We have revealed only *five* symmetry-related invariant manifolds for our model (see Fig. 1), which admit existence of localized vibrations (in general, these vibrations can be quasiperiodic). In Fig. 1, we depict the corresponding symmetry group G below the fragment of each manifold. Such information can be useful when it is necessary to enlarge this fragment because of weak decreasing of dynamical variables from the center of the manifold to its periphery.

$$Q_{3x3}^{(1)} = \begin{bmatrix} c & b & c \\ b & a & b \\ c & b & c \end{bmatrix} \qquad Q_{3x3}^{(2)} = \begin{bmatrix} d & b & d \\ c & a & c \\ d & b & d \end{bmatrix} \qquad Q_{3x3}^{(3)} = \begin{bmatrix} e & b & d \\ b & a & c \\ d & c & f \end{bmatrix}$$

$\{C_4, \sigma_x\}$ $\{\sigma_x, \sigma_y\}$ $\{\sigma_{xy}\}$

$$Q_{3x3}^{(4)} = \begin{bmatrix} e & b & f \\ c & a & d \\ e & b & f \end{bmatrix} \qquad Q_{3x4}^{(5)} = \begin{bmatrix} c & b & -b & -c \\ b & a & -a & -b \\ c & b & -b & -c \end{bmatrix}$$

$\{\sigma_y\}$ $\{\sigma_x P, \sigma_y\}$

Fig. 1. Symmetry-related invariant manifolds

2. CONSTRUCTING DISCRETE BREATHERS FOR HOMOGENEOUS POTENTIALS

Each invariant manifold depends on a number of arbitrary parameters (a, b, c, …). To construct discrete breather we must find such values of these parameters which lead to a *time-periodic* vibration when they are used as initial values for integrating dynamical equations of the considered model.

In the present paper, we use a model which differ from that of [3] by the presence of the on-site potential. Dynamical equations of the model for homogeneous potential of *m* degree read

$$\ddot{q}_{i,j} + \gamma q_{i,j}^{\,m-1} = (q_{i,j+1} - q_{i,j})^{m-1} - (q_{i,j} - q_{i,j-1})^{m-1} + (q_{i+1,j} - q_{i,j})^{m-1} - (q_{i,j} - q_{i-1,j})^{m-1} \qquad (3)$$

$$i = 1..N,\ j = 1..M.$$

The periodic boundary conditions are supposed to be hold.

The specific structure of Eqs. (3) admits the space-time separation and, as a consequence, we can treat DBs for the case of homogeneous potential in terms of localized nonlinear normal modes by Rosenberg [13]. To this end, we assume that

$$q_{i\,j}(t) = k_{i\,j} \cdot f(t), \qquad (4)$$

where i=1..N, j=1..M, while k_{ij} are constant coefficients.

Substituting the ansatz (4) into differential equations (3), we reduce them to a number of nonlinear algebraic equations, which determine the spatial profile of DB, and one ("governing") differential equation, which determines time-dependence of all the dynamical variables $q_{ij}(t)$. This time-dependence is described by the single function $f(t)$.

If we now take into account particular structures of the invariant manifolds depicted in Fig. 1, the number of unknown coefficients k_{ij} will be equal to the number of the manifold arbitrary parameters a, b, c,… minus one, since one of these parameters can be assumed equal to 1. For the invariant manifold $Q_{3x3}^{(1)}$ we can write the following algebraic equations (here we assume a=1)

$$\begin{aligned} b[-\gamma + 4(b-1)^{m-1}] &= -\gamma b^{m-1} + 2(c-b)^{m-1} + (1-b)^{m-1}, \\ c[-\gamma + 4(b-1)^{m-1}] &= -\gamma c^{m-1} + 2(b-c)^{m-1} \end{aligned} \qquad (5)$$

while the governing equation takes the form

$$\ddot{f}(t) + [\gamma - 4(b-1)^{m-1}]\, f(t)^{m-1} = 0. \qquad (6)$$

Demanding |a|>|b|>|c|, we restrict ourselves by localized breather profile.

For the homogeneous potential of m=4 degree, we have obtained the following breather spatial profile on the invariant manifold $Q_{3x3}^{(1)}$ for γ=0: a=1, b=0.25439, c=0.00439.

The time-dependence of the corresponding breather solution is determined by the governing equation

$$\ddot{f}(t) + p^2 \cdot f(t)^3 = 0, \qquad p^2 = -4(b-1)^3 \qquad (7)$$

with analytical solution of the form

$$f(t) = cn(\omega t, \frac{1}{\sqrt{2}}), \qquad \omega = p \cdot f(0). \qquad (8)$$

Proceeding in such manner, we can construct discrete breathers for the analyzed invariant manifold $Q_{3x3}^{(1)}$ for *different* values of γ which determines relative strength of the on-site and inter-site parts of the potential energy of the considered dynamical system. We plot the functions $b=b(\gamma)$ and $c=c(\gamma)$ in Fig. 2A (note that a=1 is not depicted in this figure).

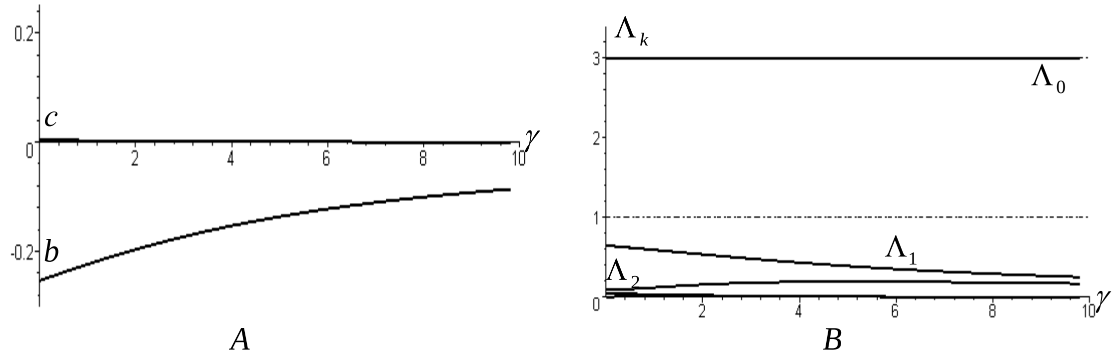

Fig. 2. DBs for invariant manifold $Q_{3x3}^{(1)}$ *: A) profiles; B) stability indicators.*

3. STABILITY OF DISCRETE BREATHERS FOR HOMOGENEOUS POTENTIALS

DBs represent time-periodic dynamical regimes and, therefore, standard Floquet method can be used for analyzing their stability. However, for the models with homogeneous potentials, more simple method was developed in [14, 15]. In this case, equations, linearized near the exact breather solution, can be written as follows:

$$\ddot{\delta} = g(t)A \cdot \delta \tag{9}$$

Here $\delta(t)$ is a vector perturbation of the exact breather solution, $g(t) = -(m-1) f^{m-2}(t)$ is a time-periodic function determined by *f(t)* from Eq. (6), while *A* is a certain time-independent matrix.

Because of the above structure of the variational equations (9), one can reduce the matrix *A* to a diagonal form with an appropriate orthogonal transformation in δ-space. As a result, the coupled equations (9) turn out to be split into independent scalar equations of one and the same form. For example, for the homogeneous potential of *m*=4 degree, we obtain from Eq. (3) $N \times M$ independent Lamé's equations

$$\ddot{z}_{i,j} + \Lambda_k f^2(t) \cdot z_{i,j} = 0, \tag{10}$$

Coefficients $\Lambda_k = \dfrac{3\lambda_k}{p^2}$, which we call *stability indicators*, depend on eigenvalues λ_k of the matrix *A*.

On the other hand, the boundaries of the regions of stable and unstable motion for the Lamé equation in the form (10) turn out to be integer numbers $\dfrac{n(n-1)}{2}$ where n=1, 2, 3, … . Moreover, the stable regions of zero solution of Eqs. (10) satisfy the condition $\Lambda_k \in [0;1], \quad [3;6], \quad [10;15], \ldots$ Therefore, the breather solution will be stable if *all* Λ_k fall in the above stability intervals.

In Fig. 2B, we plot the functions $\Lambda_k(\gamma)$ for all eigenvalues of the matrix *A* (note that some of them are equal to each other and some are too small to be visible in this figure). From Fig. 2B, we see that DBs constructed on the invariant manifold $Q_{3x3}^{(1)}$ for m=4 are stable for *all values* of γ.

Note that $\Lambda_0 = 3$ lies exactly on the lower boundary of the *second* stability region for all the values of the parameter γ. Since the corresponding eigenvector of the matrix *A* coincides with the breather's profile, the indicator Λ_0 does not affect the breather stability. Below, we don't depict such marginal stability indicators in all figures similar to Fig. 2B.

Slightly another situation takes place for the breather stability analysis in cases with homogeneous potential of higher than 4 degrees (m>4). Indeed, the coupled variational equations (9) can also be split into independent equations:

$$\ddot{z}_{i,j} + \Lambda_k f^{m-2}(t) \cdot z_{i,j} = 0. \tag{11}$$

Unlike the Lamé-case, we do not know any analytical results for detecting stability-instability regions for Eq. (11). However, our numerical experiments lead to very interesting result. It turns out that for homogeneous potential of m degree the first stability regions for zero solution of Eq. (11) are

$$\Lambda_k \in [0;1], \quad [\mathrm{m}-1;\mathrm{m}+2], \quad [3\mathrm{m}-2;3\mathrm{m}+3], \ldots \tag{12}$$

The regions of instability lie between the above listed stable regions. We believe that these numerical finding can be proved rigorously by some analytical method, but we have not such a proof at the present time.

In any case, we can detect stability of DBs in the lattices with homogeneous potentials of degree *m* using the condition that all the indicators Λ_k in Eq. (11) lie in the stability regions (12).

We plot Λ_k as functions of γ for DBs constructed on invariant manifold $Q_{3x3}^{(1)}$ for *m*=6 and *m*=8 in Fig. 3. As one can see from this figure for *m*=8, the discrete breather, being stable for $\gamma > \gamma_c$, becomes unstable for $0 \le \gamma < \gamma_c$, where $\gamma_c = 5.8848$.

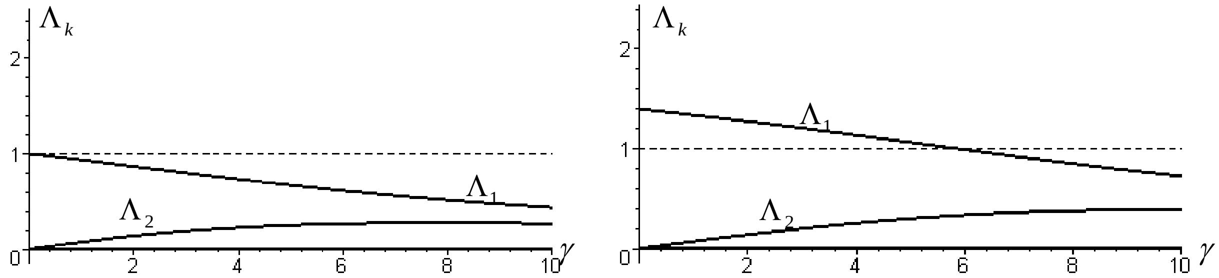

Fig. 3. Stability indicators Λ_k for DBs associated with invariant manifold $Q_{3x3}^{(1)}$ for m=6 (left) and m=8 (right).

Note that for m=6 discrete breather possesses a tendency to lose its stability with decreasing γ, because $\Lambda_1 \to 1$ for $\gamma \to 0$.

4. RESULTS AND DISCUSSION

Above we have discussed DBs constructed on the invariant manifold $Q_{3x3}^{(1)}$ which depends on three arbitrary parameters *a, b, c* (see Fig. 1). If we enlarge 3×3 lattice fragment, corresponding to this manifold, with the aid of the symmetry group $\{C_4, \sigma_x\} = C_{4v}$, new arbitrary parameters appear. However, the numerical values of these additional parameters which are obtained as a result of constructing DBs on the considered manifold turn out to be very small because of the breather's strong localization and, therefore, they can be neglected. Because of this reason, we don't indicate below the size of lattice fragments near the manifolds depicted in Fig. 1 and refer to them as $Q^{(j)}$ (*j*=1..5).

We have already considered DBs on the $Q^{(1)}$. Let us discuss these dynamical objects associated with other invariant manifolds.

1. The manifolds $Q^{(2)}$ and $Q^{(3)}$ with symmetry groups $\{\sigma_x, \sigma_y\} = C_{2v}$ and $\{\sigma_{xy}\} = C_s$, respectively.

- m=4.

  We did not obtain any DBs with such symmetries. For example, for the manifold $Q^{(2)}$, we found only Rosenberg mode with *a*=*b* which corresponds to the more symmetric manifold $Q^{(1)}$ with the point group C_{4v}.
- m=6.

  DBs exist only for $\gamma \in [0; \gamma_c = 0.0476]$. For $\gamma > \gamma_c$ these breathers transform into DBs associated with the manifold $Q^{(1)}$. Moreover, these breathers turn out to be *unstable*.
- m=8.

  DBs exist only for $\gamma \in [0; \gamma_c = 5.8848]$ and they are unstable.

2. Invariant manifold $Q^{(4)}$ with symmetry group $\{\sigma_y\} = C_s$.

- m=4.

  There are no DBs associated with this manifold.
- m=6.

  DBs exist only in two intervals $\gamma \in [0; 0.0476]$ and $\gamma \in [0.0484; 3.8716]$. These DBs are *unstable*.
- m=8.

  DBs exist only in two intervals $\gamma \in [0; 5.8848]$ and $\gamma \in [5.8862; 17.9037]$. These DBs are *unstable*.

3. Invariant manifold $Q^{(3)}$ with symmetry group $\{\sigma_x P, \sigma_y\}$.

Note that only this manifold is associated with symmetry group whose some elements contain the operator P changing signs of all the displacements without their transposition. As a consequence, arbitrary parameters *a, b, c* enter this manifold with different signs.

- m=4.

DBs exist for *all values* γ, but they are *unstable.*

- m=6 and m=8.

DBs exist for *all values* γ. They are stable only for $\gamma \in [0; 3.9535]$ and $\gamma \in [0; 18.0117]$, respectively.

Up to this point, we have associated DBs in systems with homogeneous potential with localized nonlinear normal modes by Rosenberg. However, for more general potentials, such type of modes, as a rule, *don't* exist, while DBs can exist. In this case, DBs represent localized vibrations for which, unlike the Rosenberg modes, displacements of different particles are described by *different time-dependent functions*, although their vibrational periods are equal or commensurate [15].

In conclusion, let us emphasize that symmetry-related invariant manifolds, which are found in the present paper, can be used for constructing DBs in square lattice with *arbitrary* potentials even those who prevent existence of the Rosenberg modes. Certainly, in this case, we must apply more general methods for search breather solution described in [16] or the method of pair synchronization [15].

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