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FAMILIES OF PERIODIC SOLUTIONS OF HILL'S PROBLEM

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ABSTRACT

Generating periodic solutions of Kepler problem in case of Hill's perturbation are investigated. Existence of two kinds of generated solutions by means of normal forms is proved.

INTRODUCTION

An efficient method of periodic orbits search is the method of generating solutions which was independently developed by A.D. Bruno [1] and M. Henon [2]. This method suggests to consider a nonintegrable problem as a perturbation of integrable one, therefore it makes possible to isolate such periodic solutions of unperturbed problem that may be continued into the periodic solution of nonintegrable one.

We study generating solutions of Kepler problem in synodical coordinates which is perturbed with polynomial. This perturbation allows investigating families of period solutions of Hill's problem by periodic orbits of Kepler problem. The majority of known families of periodic solutions of Hill's problem were studied as continuation of generating solutions of a linear Hamiltonian system perturbed with singular function [3]. We propose to study families of periodic solutions of Hill's problem using the continuation of generating solutions of Kepler problem. These generating solutions are obtained with the help of approach described in [1, Chapter VII].

1. HILL'S PROBLEM EQUATIONS AND THEIR PROPERTIES

Hill's problem is a limiting case of the well-known restricted three-body problem (RTBP). It describes the dynamics of a massless body (satellite) in the vicinity of the minor primary (Earth). Hill's problem equations may be obtained in several ways but usually RTBP equations are transformed with so called Hill's transformation. Hill's transformation consists of two linear transformations. The first is a shift of coordinate origin at the position of the minor primary and the second is a special coordinate scaling. In the planar case Hill's transformation is written in form

$$\begin{aligned} x_1 &= \mu - 1 + \mu^{1/3} X_1, & y_1 &= \mu^{1/3} Y_1, \\ x_2 &= \mu^{1/3} X_2, & y_2 &= \mu - 1 + \mu^{1/3} Y_2 \end{aligned} \qquad (1)$$

The easier way to obtain Hill's problem equations of motion is to apply Hill's transformation (1) to Hamiltonian of RTBP with generating function [4] or using Power Geometry technique [5].

Hamiltonian of the planar circular Hill's problem is

$$H = \frac{1}{2}(y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - \frac{1}{r} - x_1^2 + \frac{1}{2} x_2^2, \text{ where } r = \sqrt{x_1^2 + x_2^2}, \qquad (2)$$

here x – vector of canonic coordinates, y – vector of canonically conjugate momenta. Canonical equations of motion are invariant under finite group of transformations of order 4

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$$\Sigma_1: (t, x_1, x_2, y_1, y_2) \to (-t, x_1, -x_2, -y_1, y_2)$$
$$\Sigma_2: (t, x_1, x_2, y_1, y_2) \to (-t, -x_1, x_2, y_1, -y_2) \quad (3)$$

and their composition $\Sigma_1 \circ \Sigma_2$. In the configuration space these transformations act as axial symmetries relative to abscissa axis, ordinate axis and the coordinate origin correspondingly. The presence of these symmetries leads to the fact that all the families of periodic solutions form four classes.

Nonsymmetrical orbits. Three more orbits correspond to each such orbit with the same period and initial conditions which are obtained with mentioned above transformations.

Σ_1-symmetric orbits. One more orbit corresponds to each such orbit which is symmetrical with respect to the ordinate axis.

Σ_2-symmetric orbits. One more orbit corresponds to each such orbit which is symmetrical with respect to the abscissa axis.

Double symmetric orbits. These orbits are symmetrical with respect to coordinate axes to themselves.

Hill's problem equations of motion possess the single first integral of motion which called Jacobi integral:

$$C = 3x_1^2 + 2/r - \dot{x}_1^2 + \dot{x}_2^2 = -2H \quad (4)$$

and therefore it is nonintegrable problem [2].

Hill's problem has complex set of one-parameter families of periodic solutions which can be continued into the periodic solutions of the RTBP. There are a lot of various applications of Hill's problem [4].

2. HILL'S PROBLEM AS A SINGULARLY PERTURBED LINEAR HAMILTONIAN SYSTEM

Canonical change of variables $x = \sqrt{|C|}X,\ y = \sqrt{|C|}Y$ allows to write Hamiltonian (2) of Hill's problem as a perturbation of linear Hamiltonian system H_0 called *Henon's intermediate problem* with singular function R: $H = H_0 + \varepsilon R$, where

$$H_0 = \frac{1}{2}(Y_1^2 + Y_2^2) + X_2Y_1 - X_1Y_2 - X_1^2 + \frac{1}{2}X_2^2 = \frac{1}{2},\ R = -\frac{1}{r},\ \varepsilon = \frac{1}{|C|^{3/2}}. \quad (5)$$

Normal form $H_0 = (Q_1^2 + P_1^2)/2 - 3/2P_2^2$ of Hamiltonian H_0 is obtained with the help of generating function $S_2(X,P) = X_1(P_2 - X_2) + P_2(X_2 - 2P_1)$, which produces canonical variable change $X_1 = Q_1 + 2P_2, X_2 = Q_2 + 2P_1, Y_1 = -P_1 - Q_2, Y_2 = -P_2 - Q_1$.

Canonical equations of motions defined by Hamiltonian H_0 has one-parameter family of periodic orbits with period $T_0 = 2\pi$ in form of ellipses with semi-axes 1 and 2 and with the center at point $(0, Q_2^0)$. One more canonical variable change $Q_1 = \sqrt{2L}\cos\varphi,\ P_1 = \sqrt{2L}\sin\varphi$ of the first pair of conjugated coordinates together with time inversion $t = -\tau$ allow to write Hamiltonian H_0 in the form $H_0 = -L + 3/2P_2^2$. So one-parameter family of periodic solutions is defined by following values

$$\varphi = -\tau,\ L = 1/2,\ Q_2 = Q_2^0,\ P_2 = 0 \quad (6)$$

The generating solutions of unperturbed problem satisfy condition $\partial[R]/\partial Q_2^0 = 0$, where square brackets mean averaging along solution (6) (see Chapter VII, §1 [1]):

$$[R](Q_2^0) = \frac{1}{2\pi}\int_0^{2\pi} R(Q_2^0,\tau)d\tau = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\tau}{\sqrt{Q_2^0(Q_2^0 - 4\sin\tau) + 3\sin^2\tau + 1}} \quad (7)$$

Function (7) has the only extremum at the point $Q_2^0 = 0$, therefore the only regular generating solution of Henon's intermediate problem is an ellipse (6) with center at the origin. This solution is easily identified with retrograde single-turn orbits of family ***f*** originally computed by J. Jackson (1913) and T. Matukuma (1932) (see [3]). The first order correction to the period of generating solution is computed by formula $T = T_0 + \varepsilon T_0 \partial[R]/\partial L + O(\varepsilon^2)$, where averaging is calculated along the generating solution (6). Finally, the asymptotic of the period of family ***f*** is of form

$$T = 2\pi - 2K(\sqrt{3}/2)/|C|^{3/2} + O(|C|^{-3}), \quad (9)$$

here K is complete elliptic integral of the first kind. It is necessary to notice that the first order corrections to the period and initial conditions of the family $\boldsymbol{f}$ were calculated in [6] but formulas mentioned in the pages 345-346 are wrong.

3. HILL'S PROBLEM AS A REGULARLY PERTURBED KEPLER PROBLEM

Hill's problem may be presented in a form of perturbed Kepler problem in synodical coordinates: $\widetilde{H} = H_K + \varepsilon R$, where

$$H_K = \frac{1}{2}(y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - \frac{1}{r}, R = \frac{1}{2}x_2^2 - x_1^2 \quad (10)$$

One can obtain Hamiltonian of Kepler problem in synodical coordinates for $\varepsilon = 0$ and Hamiltonian of Hill's problem for $\varepsilon = 1$. Let’s consider three main types of periodic solutions of Kepler problem: stationary points, circular orbits and elliptic orbits. It should be noticed that periodic solutions of Kepler problem in inertial (sidereal) coordinates are preserved in synodical coordinates in two cases: either sidereal orbit is circular or period of siderial orbit is commensurable with 2π. We investigate only last case in this paper.

3.1 Perturbation of the stationary points

Stationary points of canonical equations of motion of Kepler problem form the unit circle $r = 1$ on the plane *XOY*, but perturbation R from (10) destroys it and leaves two pairs of stationary points. First pair of stationary points with coordinates $(\pm(1 + 2\varepsilon)^{-1/3}, 0)$ lays on X-axis and corresponds to the well known collinear Lagrange libration points. Second pair of stationary points with coordinates $(0, \pm(1 - \varepsilon)^{-1/3})$ lays on Y-axis and tends to infinity while $\varepsilon \to 1$. It is possible to expect that periodic orbits situated outside the unit circle of stationary points will be destroyed at Hill's perturbation (10).

3.2 Perturbation of elliptic orbits

Let's write Hamiltonian of generating problem in modified Delaunay coordinates [7, Chapter 7] which provide the normal form of Hamiltonian H_K at vicinity of integral manifold of direct and retrograde elliptic orbits $\mathcal{D}d_N$ and $\mathcal{D}r_N$ [1, Chapter 7, §2]. The first approximation of normal form of Hamiltonian $\widetilde{H}$ is obtained by averaging of perturbation R (see Chapter 3 [8]). So Delaunay variables (l, g, L, G) provide the following form of the functions (10)

$$H_K = -G - \frac{1}{L^2}, R = \frac{1}{2}r^2 - \frac{3}{2}r^2 \cos^2 h \quad (11)$$

here r and h are polar coordinates. Elliptic orbites are defined by the following values of Delaunay variables in case $\varepsilon = 0$:

$$L = \sqrt{a}, G = \varepsilon'\sqrt{a(1 - e^2)}, l = Nt, g = \theta - t \quad (12)$$

here a is a semimajor axis of the orbit, e is its eccentricity, $N = a^{-3/2}$ is mean motion, θ is pericentre angle and $\varepsilon' = \pm 1$ specifies the direction of the motion. The value $\varepsilon' l$ is mean anomaly which defines the position of the point on the elliptic orbit at specified moment of time, and the value g defines the motion of the orbit pericentre, which uniformly rotate anticlockwise in synodical coordinates. Finally, L is defined by energy integral and G is area integral of Kepler problem.

Let’s consider an integral manifold $\mathcal{M} \in \mathcal{D}d_N \cup \mathcal{D}r_N$ which consists of periodic orbits with rational mean motion $N = 1 + q/p$ and with period $T_0 = 2\pi p$. Hamiltonian $\widetilde{H}$ be a function of angle variable θ and generalized momentum G along manifold $\mathcal{M}$. Generating orbits are specified by condition

$$\partial[R(\theta, G)]/\partial\theta = 0 \quad (13)$$

according to formula (1.15) [1, Chapter 7], where square brackets means averaging along the elliptic orbit (12). The averaging of function R is carried out over mean anomaly l, so

$[R] = \frac{1}{2\pi}\int_0^{2\pi(p+q)} R(l)N^{-1}dl$. Function R may be written in form $R = R_1 + R_2$, where $R_1 = -r^2/4$ and $R_2 = -(3/2)\, r^2 \cos 2h$. Since function R_1 does not depend on θ so condition (13) becomes trivial for its averaging. Function R_2 may be written in the form of Fourier series over mean anomaly with coefficients depending on Bessel functions of the first kind [9].

The following statement is proved:

$$[R_2] = \begin{cases} -\dfrac{3\sqrt[3]{p}}{2e} \cos 2\theta\, S_{2p}(\varepsilon', e)\ if\ p+q=1, \\ -\dfrac{3\sqrt[3]{p}}{\sqrt[3]{4e}} \cos 2\theta\, S_p(\varepsilon', e)\ if\ p+q=2, \\ 0\ if\ p+q>2, \end{cases}$$

where $S_p(\varepsilon', e) = (1-e^2)\, J_p'(pe) - \frac{2-e^2}{pe} J_p(pe) + \varepsilon' e\sqrt{1-e^2}\, J_p''(pe)$ and J_p is Bessel function of the first kind. Function $S_p(\varepsilon', e)$ was numericaly investigated for $p \leq 1000$ and it has following properties:

1. $S_p(\varepsilon', e)$ is analytical in the interval $0 \leq e < 1$,
2. $S_p(\varepsilon', 1) = -J_p(p)/p < 0$,
3. Equation $S_p(\varepsilon', e) = 0$ has the unique root e_p^* in the interval $0 < e < 1$ if $\varepsilon' = 1$ and $p > 1$.

Thus there are two main families of generating solutions satisfying condition (13).

Case 1. Symmetric orbits with $\theta = k\pi/2$, $k = 0,1,2,3$ and $0 < e < 1$.

If $p + q = 1$ then evenness of integers p and q is different and there are two families of Σ_1-symmetric orbits corresponding $\theta = 0, \pi$ and two families of Σ_2-symmetric orbits corresponding $\theta = \pi/2, 3\pi/2$. Some orbits for this case are shown on fig. 1.

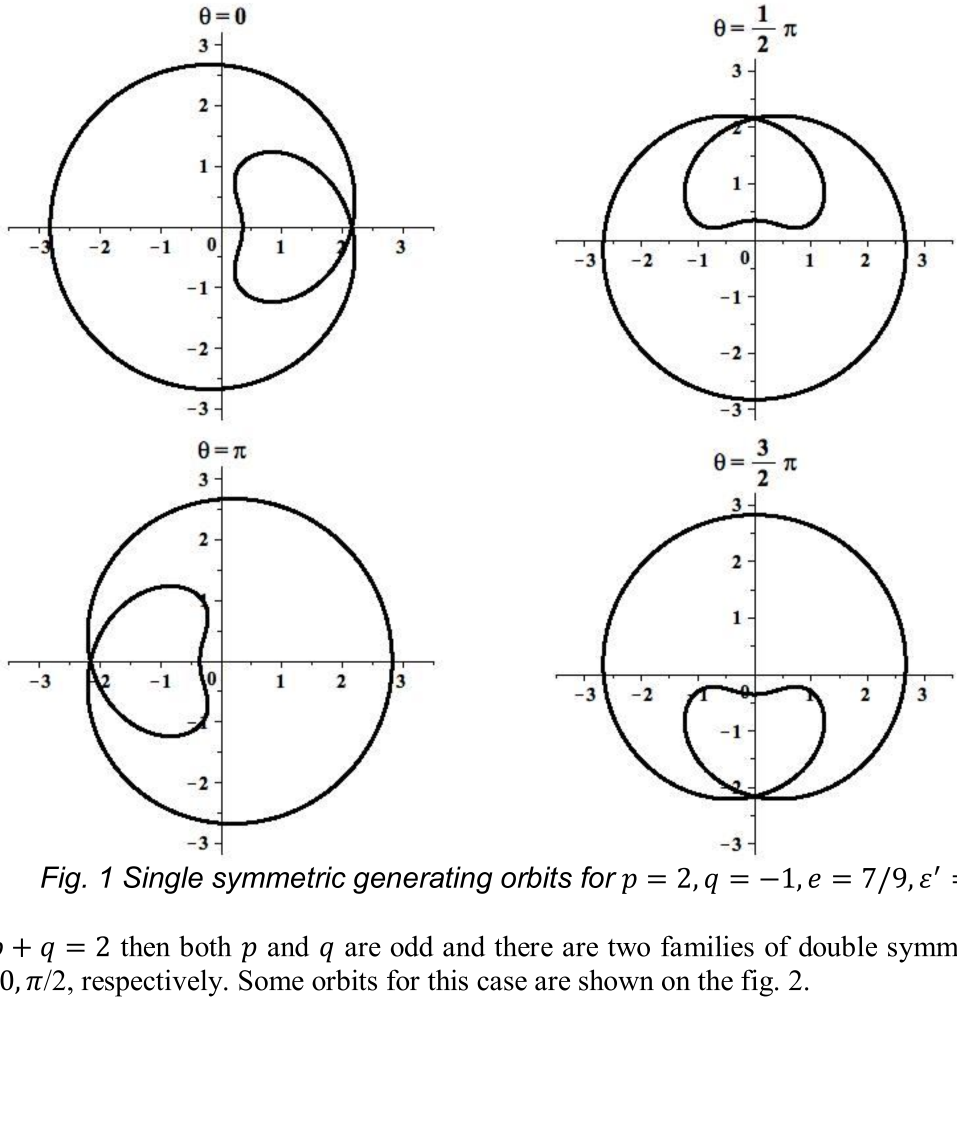

Fig. 1 Single symmetric generating orbits for $p = 2, q = -1, e = 7/9, \varepsilon' = 1$.

If $p + q = 2$ then both p and q are odd and there are two families of double symmetric orbits with $\theta = 0, \pi/2$, respectively. Some orbits for this case are shown on the fig. 2.

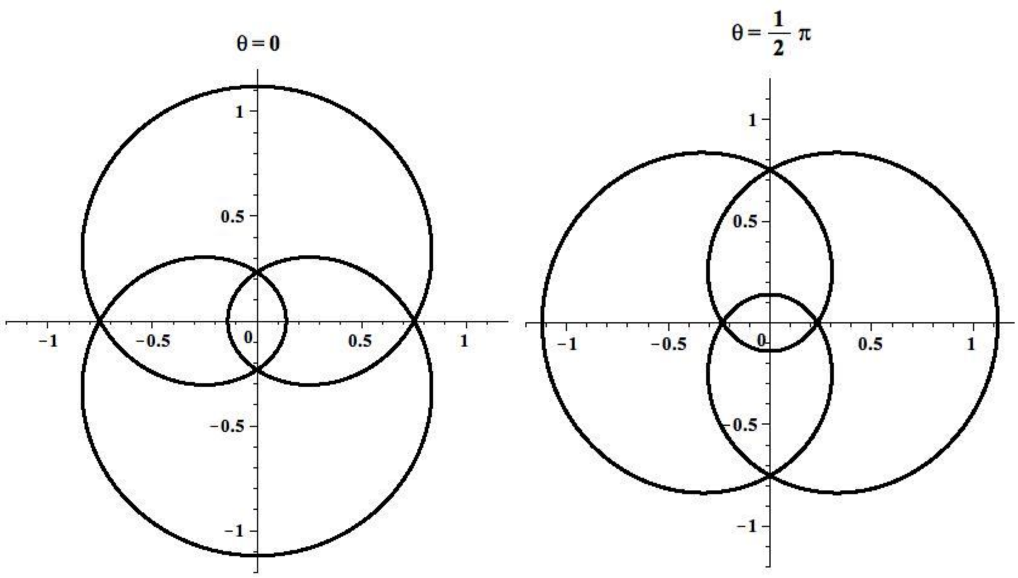

Fig. 2 Double symmetric generating orbits for $p = 1, q = 1, e = 7/9, \varepsilon' = -1$.

Case 2. Asymmetric orbits with $\theta \neq k\pi/2$, $k = 0,1,2,3$, $p > 1$, $\varepsilon' = 1$ and $e = e_p^*$. Approximate values of eccentricity e corresponding to asymmetric generating orbits for $p = 2, \ldots, 10$ are listed in table 1.

Table 1 Critical value of eccentricity for asymmetric generating orbits

N	p	e_p^*
1	2	0.75822858044480
2	3	0.85254323545950
3	4	0.89215536030911
4	5	0.91403781912693
5	6	0.92797039943780
6	7	0.93765362124966
7	8	0.94479628596787
8	9	0.95029677982004
9	10	0.95467290426209

Asymetric generating orbits are shown in fig. 3. Left picture corresponds the case $p = 2, q = -1$ and right picture corresponds the case $p = 3, q = -1$.

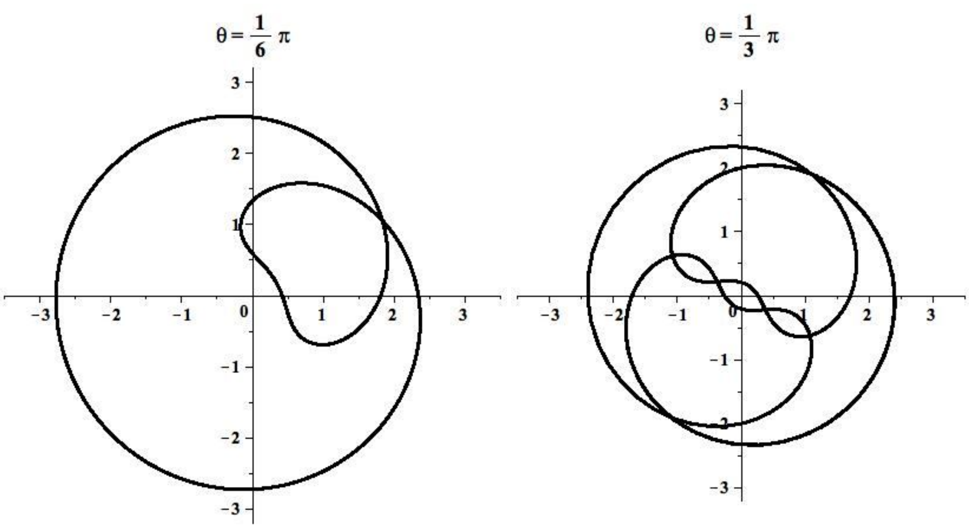

Fig. 1 Asymmetric generating orbits with $p = 2, q = -1$ *(left) and* $p = 3, q = -1$ *(right).*