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NONLINEAR ANALYSES OF THE LAMINATED PLATES OF THE SYMMETRIC STRUCTURE SUBJECTED TO STATIC IN THE PLANE FORCES

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ABSTRACT

The numerically-analytical method of nonlinear vibration research for laminated plates loaded by static in-plane force is proposed. The joint use of the R-functions and variational methods allows apply the offered approach to a plate with a complex form and different types of boundary conditions.

INTRODUCTION

The analysis of the geometrically nonlinear vibrations of composite plates and shells have received an exceptional interest in literature due to wide application of laminated plates for modeling elements in modern structures. Usually such elements have a different shape and therefore the study of dynamical behavior of these elements is a very difficult mathematical problem. In this work we propose effective approach based on using variational methods and the R-functions theory (RFM) in order to carry out the nonlinear analysis of laminated plates with an arbitrary planform and different boundary conditions, which are subjected to static load in the middle plane. Formerly this approach was successfully used for orthotropic plates [7] and for the investigation of free nonlinear vibrations of laminated plates and shells [5, 8-10]. The action of static load in the middle plane leads to the deformation of plate and affects the dynamic behavior. It should be noted that the study of plate vibrations subjected to static load is also important because it is part of the dynamic analysis of plates with periodic load, dynamic instability and parametric vibrations [2].

The proposed method is numerically implemented in the system POLE-RL and is illustrated by some examples.

1. FORMULATION OF THE PROBLEM

Let us consider free geometrically nonlinear vibrations of laminated plates of a symmetric structure in relation to the middle plane, which is subjected to a static load in its plane. It is assumed that the delamination of the layers is absent. The mathematical formulation of the problem is made in the framework of the classical theory based on the Kirchhoff – Leave hypotheses. Let us consider the movement equations in operator form [1, 6]:

$$L_{11}(u)+L_{12}(v)=-Nl_1(w), \tag{1}$$

$$L_{21}(u)+L_{22}(v)=-Nl_2(w), \tag{2}$$

$$L_{33}(w)=-Nl_3(u,v,w)-m_1\frac{\partial^2 w}{\partial t^2}, \tag{3}$$

where u,v,w are displacements of the plate in directions Ox, Oy and Oz respectively. In expressions (1)-(3) the differential operators L_{ij}, Nl_i $i,j=1,2,3$ are defined as follows:

$$L_{11}=C_{11}\frac{\partial^2}{\partial x^2}+2C_{16}\frac{\partial^2}{\partial x\partial y}+C_{66}\frac{\partial^2}{\partial y^2},$$

$$L_{22}=C_{66}\frac{\partial^2}{\partial x^2}+2C_{26}\frac{\partial^2}{\partial x\partial y}+C_{22}\frac{\partial^2}{\partial y^2},$$

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$$L_{12}=L_{21}=C_{16}\frac{\partial^2}{\partial x^2}+\left(C_{12}+C_{66}\right)\frac{\partial^2}{\partial x\partial y}+C_{26}\frac{\partial^2}{\partial y^2},$$

$$L_{33}=D_{11}\frac{\partial^4}{\partial x^4}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4}{\partial y^2\partial x^2}+4D_{16}\frac{\partial^4}{\partial x^3\partial y}+4D_{26}\frac{\partial^4}{\partial y^3\partial x}+D_{22}\frac{\partial^4}{\partial y^4},$$

$$Nl_1=\frac{\partial}{\partial x}\left\{\frac{1}{2}C_{11}\left(\frac{\partial}{\partial x}\right)^2+\frac{1}{2}C_{12}\left(\frac{\partial}{\partial y}\right)^2+C_{16}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right\}+\frac{\partial}{\partial y}\left\{\frac{1}{2}C_{16}\left(\frac{\partial}{\partial x}\right)^2+\frac{1}{2}C_{26}\left(\frac{\partial}{\partial y}\right)^2+C_{66}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right\},$$

$$Nl_2=\frac{\partial}{\partial x}\left\{\frac{1}{2}C_{16}\left(\frac{\partial}{\partial x}\right)^2+\frac{1}{2}C_{26}\left(\frac{\partial}{\partial y}\right)^2+C_{66}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right\}+\frac{\partial}{\partial y}\left\{\frac{1}{2}C_{12}\left(\frac{\partial}{\partial x}\right)^2+\frac{1}{2}C_{22}\left(\frac{\partial}{\partial y}\right)^2+C_{26}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right\},$$

$$Nl_3=N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}.$$

Here N_x, N_y, N_{xy} – normal and tangential forces in the middle plane, which are determined for multilayer plates by known formulas shown below in the matrix form [1]:

$$\overrightarrow{N}=(N_x,N_y,N_{xy})^T=\mathbf{C}\cdot\vec{\varepsilon},\ \text{where}\ \mathbf{C}=\begin{pmatrix}C_{11}&C_{12}&C_{16}\\C_{12}&C_{22}&C_{26}\\C_{16}&C_{26}&C_{66}\end{pmatrix},\ \vec{\varepsilon}=(\varepsilon_x,\varepsilon_y,\varepsilon_{xy})^T, \tag{4}$$

In these formulas the deformation components ε_x, ε_{xy}, ε_y, are defined as

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2;\ \varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x};\ \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2;$$

The values m_1 and C_{ij},D_{ij} $(ij=11,22,12,16,26,66)$, are defined as follows:

$$m_1=\sum_{s=1}^{N}\int_{h_s}^{h_{s+1}}\rho_s dz, \tag{5}$$

$$\left(C_{ij},D_{ij}\right)=\sum_{s=1}^{N}\int_{h_s}^{h_{s+1}}B_{ij}^{(s)}\left(1,z^2\right)dz. \tag{6}$$

In general, when the anisotropy axes do not coincide with the axes *Ox* and *Oy* elastic constants of the s-layer $B_{ij}^{(s)}(i,j=1,2,6)$ are expressed through the elastic constants of the initial system $\tilde{B}_{ij}^{(s)}(i,j=1,2,6)$ by the known formulas [1].

The system of equations is supplemented by boundary conditions, the expressions of which are determined by the way of fixing and loading of the plate boundary.

On the loaded part of the border the boundary conditions for the displacements in the plane are defined as

$$\begin{aligned}N_n&=-p,\\T_n&=0.\end{aligned} \tag{7}$$

where N_n,T_n - normal and tangential forces in the middle plane. Let us present them as follows:

$$N_n=N_n^{(L)}+N_n^{(D)},$$

$$N_n^{(L)}=\frac{\partial u}{\partial x}\left(C_{11}l^2+C_{12}m^2+2C_{16}lm\right)+\frac{\partial v}{\partial y}\left(C_{12}l^2+C_{22}m^2+2C_{26}lm\right)+$$

$$+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\right)\left(C_{16}l^2+C_{26}m^2+2C_{66}lm\right),$$

$$N_n^{(D)} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \left(C_{11}l^2 + C_{12}m^2 + 2C_{16}lm\right) + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \left(C_{12}l^2 + C_{22}m^2 + 2C_{26}lm\right) +$$

$$+ \frac{\partial w}{\partial y} \cdot \frac{\partial w}{\partial y}\left(C_{16}l^2 + C_{26}m^2 + 2C_{66}lm\right),$$

$$T_n = T_n^{(L)} + T_n^{(D)},$$

$$T_n^{(L)} = \frac{\partial u}{\partial x}\left(C_{11}(l^2 - m^2) + (C_{12} - C_{11})lm\right) + \frac{\partial v}{\partial y}\left(C_{26}(l^2 - m^2) + (C_{22} - C_{12})lm\right) +$$

$$+ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)\left(C_{66}(l^2 - m^2) + (C_{26} - C_{16})lm\right),$$

$$T_n^{(D)} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \left(C_{16}(l^2 - m^2) + (C_{12} - C_{11})lm\right) + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \left(C_{26}(l^2 - m^2) + (C_{22} - C_{12})lm\right) +$$

$$+ \frac{\partial w}{\partial y} \cdot \frac{\partial w}{\partial y}\left(C_{66}(l^2 - m^2) + (C_{26} - C_{16})lm\right).$$

2. METHOD OF SOLUTION

The proposed method consists of several stages.

1st stage. To determine the subcritical state of the plate it is necessary to find the functions that satisfy the following equations

$$L_{11}u_1 + L_{12}v_1 = 0, \quad L_{21}u_1 + L_{22}v_1 = 0 \tag{8}$$

and non-homogeneous boundary conditions

$$N_n^{(L)}(u_1, v_1) = -p, \quad T_n^{(L)}(u_1, v_1) = 0.$$

It is impatient to note that this problem can be regarded as a plane problem of elasticity theory which variational formulation is reduced to finding the minimum of the following functional:

$$I(u_1, v_1) = \frac{1}{2}\iint_{\Omega}(N_x^{(L)}\varepsilon_x + N_y^{(L)}\varepsilon_y + N_{xy}^{(L)}\varepsilon_{xy})d\Omega + \int_{\partial\Omega_1} N_n^{(L)}u_{1n}d\Omega_1, \tag{9}$$

where $\partial\Omega_1$ is part of the border loaded by the external forces

$$\vec{N}_n^{(L)} = (N_x^{(L)}, N_y^{(L)}, N_{xy}^{(L)})^T = \mathbf{C} \cdot \vec{\varepsilon}^{(L)},$$

here $\vec{\varepsilon}^{(L)} = (\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^T$.

On the whole this problem may be solved by RFM.

2nd stage. A linear problem of the plate vibrations compressed by static load in the middle plane may be solved by Ritz method as a result of functional minimization:

$$J = \Pi_{\max} - T_{\max}, \tag{10}$$

where $T_{\max}$ is kinetic energy of the plate and $\Pi_{\max}$ is maximum potential energy of the plate:

$$T_{\max} = \frac{m_1\omega_L^2}{2}\int_{\Omega}\left(u^2 + v^2 + w^2\right)d\Omega$$

$$\Pi_{\max} = \frac{1}{2}\iint_{\Omega}[(M_x\chi_x + M_y\chi_y + M_{xy}\chi_{xy}) +$$

$$+ p(N_x^{(L)}(u_1, v_1)(\frac{\partial w}{\partial x})^2 + N_y^{(L)}(u_1, v_1)(\frac{\partial w}{\partial y})^2 + N_{xy}^{(L)}(u_1, v_1)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y})]dxdy.$$

where ω_L is the natural frequency, corresponding to a given load *p*, M_x, M_y, M_{xy} are bending and shear moments, which are defined for multilayer symmetric plates as follows:

$$\overrightarrow{M} = (M_x, M_y, M_{xy})^T = \mathbf{D} \cdot \overrightarrow{\chi}; \text{ где } \mathbf{D} = \begin{pmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{pmatrix}, \ \overrightarrow{\chi} = (\chi_x, \chi_y, \chi_{xy})^T.$$

In these formulas the deformation components χ_x, χ_{xy}, χ_y are defined as

$$\chi_x = -\frac{\partial^2 w}{\partial x^2}; \ \chi_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}; \ \chi_y = -\frac{\partial^2 w}{\partial y^2},$$

Thus, the solution of the linear vibration problem is reduced to an eigenvalue problem with the appropriate boundary conditions.

3rd stage. Let us present unknown functions (u, v, w) in the following way:

$$\begin{gathered} w(x, y, t) = y(t) \cdot w_1(x, y), \\ u(x, y, t) = u_1(x, y) + y^2(t) \cdot u_2(x, y), \\ v(x, y, t) = v_1(x, y) + y^2(t) \cdot v_2(x, y). \end{gathered} \tag{11}$$

Here $w_1(x, y)$ is eigenfunction corresponding to the natural frequency ω_L, and (u_2, v_2) have to satisfy the non-homogeneous linear system of the differential equations:

$$\begin{gathered} L_{11}(u_2) + L_{12}(v_2) = -Nl_1(w_1), \\ L_{21}(u_2) + L_{22}(v_2) = -Nl_2(w_1), \end{gathered}$$

and the following boundary conditions:

$$\begin{gathered} N_n^{(L)}(u_2, v_2) = -N_n^{(D)}(w_1), \\ T_n^{(L)}(u_2, v_2) = -T_n^{(D)}(w_1). \end{gathered}$$

The solution of this problem may be reduced to the variational problem of the functional minimum determination:

$$I(u_2, v_2) = \frac{1}{2}\iint_{\Omega}(N_x \varepsilon_x + N_y \varepsilon_y + N_{xy}\varepsilon_{xy} - 2(Nl_1(w_1)u_2 + Nl_2(w_1)v_2))d\Omega + \\ + \int_{\partial\Omega_1} N_n^{(D)}(u_2 l + v_2 m) + T_n^{(D)}(-u_2 m + v_2 l)d\Omega_1,$$

Substituting expressions (11) into equation (3), and using the Bubnov-Galerkin method, we can obtainthe following ordinary nonlinear differential equation of the Duffing's type:

$$y_1''(t) + \omega_L^2\left(y(t) + \beta \cdot y^3(t)\right) = 0,. \tag{12}$$

where ω_L is the natural frequency of the linear plate vibration and β defined as follows:

$$\beta = -\frac{\int_{\Omega} Nl_3(u_2, v_2, w_1) \cdot w_1 d\Omega}{m_1 \Omega_L^2 \|w_1\|^2},$$

where N_x^L, N_y^L, N_{xy}^L are linear forces in the middle plate.

4th stage. The resulting differential equation (12) can be solved in different ways. We are appling the Bubnov-Galerkin method. Let us present the solution as follows

$$y(t) = A \cos \omega_N t, \tag{13}$$

where *A* is amplitude and ω_N is nonlinear vibration frequency. Applying Bubnov-Galerkin method to equation (12), we obtain the relationships between the ratio of linear and nonlinear fundamental frequencies $\nu = \omega_N / \omega_L$ and amplitude *A* as follows

$$\nu = \sqrt{1 + \frac{3}{4}\beta A^2}. \tag{14}$$

3. NUMERICAL INVESTIGATION.

Let us apply the proposed approach to the study of nonlinear vibration of the single-layer orthotropic plate (Fig. 1). Let us consider the follow boundary conditions:

$$w=0,\ \frac{\partial^2 w}{\partial n^2}=0,\ N_n=-p,\ T_n=0,\ \ (x,y)\in\partial\Omega_1\,(\Omega_1: x=\pm\frac{a}{2}),$$

$$w=0,\ \frac{\partial^2 w}{\partial n^2}=0,\ N_n=0,\ T_n=0,\ (x,y)\in\partial\Omega_2=\partial\Omega/\partial\Omega_1,$$

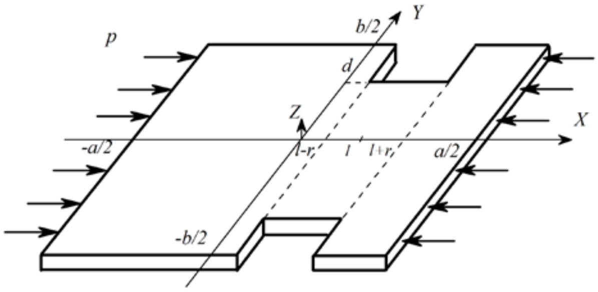

Figure 1. The single-layer orthotropic plate

For the given conditions the structure of solution [3,4] for u, v, w satisfying only the main boundary conditions takes the form of

$$u_i=P_i,\ v_i=P_{i+2},\ i=1,2,\ w=\omega\cdot P_5, \tag{15}$$

where $\omega(x,y)=0$ is the equation of the whole boundary domain. The function $\omega(x,y)$ is defined as follows

$$\omega(x,y)=f_1\wedge_0 f_2\wedge_0(\overline{f}_3\vee_0 f_4).$$

Here the functions f_1, f_2 f_3, f_4 are defined as

$$f_1=\frac{1}{a}\left(\left(\frac{a}{2}\right)^2-x^2\right)\geq 0,\ f_2=\frac{1}{b}\left(\left(\frac{b}{2}\right)^2-y^2\right)\geq 0,\ f_3=\frac{1}{2r}\left(r^2-(x-l)^2\right)\geq 0,\ f_4=\frac{1}{2d}\left(d^2-y^2\right)\geq 0.$$

The symbols $\wedge_0,\vee_0$ denote R-operations [3, 4]. In (15) P_i are indefinite components of the structure that are presented as an expansion in a series in a complete system (in this presentation power polynomials are used).

Calculations are carried out for glass – epoxy plate ($E_1/E_2=3$, $G/E_2=0.6$, $\nu_1=0.25$) with $b/a=1$, $pa^2/h^3E_2=1$. The effect of a cutouts size on amplitude-frequency characteristics has been investigated for $r/a=0.2,0.1,0.05$, $d/a=0.35,0.4,0.45$, $l/a=0.1$ (Fig. 2). In Fig. 3 amplitude-frequency characteristics depending on the disposition of cutouts are presented. For such study we use various values of ratio $l/a=0,0.1,0.2,0.3$ at fixed value of ratios $r/a=0.1$, $d/a==0.4$). The analysis of the obtained results allows draw a conclusion that the size of given plate cutouts affects the characteristics considered much stronger than its disposition.

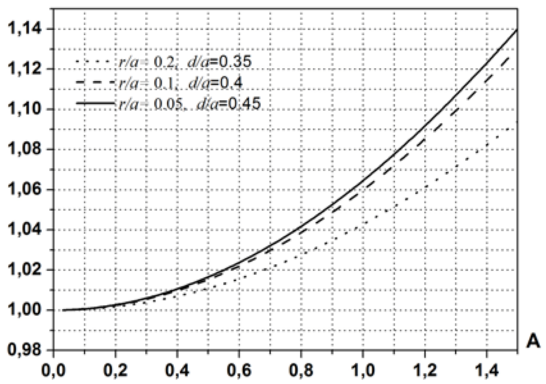

Fig. 2. Amplitude-frequency characteristics versus to cutouts size.

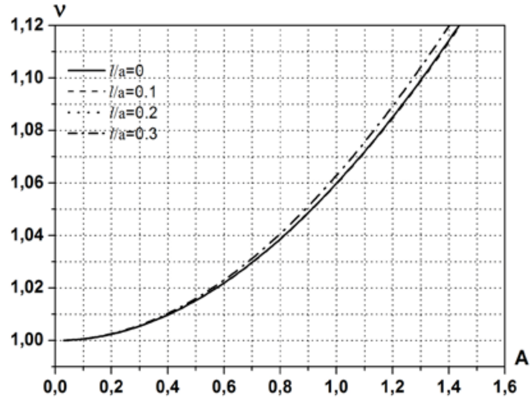

Fig. 3. Amplitude-frequency characteristics versus to cutouts disposition.

CONCLUSIONS

The method of nonlinear vibration research of in-plane loaded laminated plates with a complex form is proposed. Due to the application of R-function theory in combination with variational methods the investigation of the movement equation is reduced to studying ordinary differential equation of the Duffing type. Using the offered method and the created software the dynamic behavior of plate with cutouts subjected by static load is studied. The effect of cutout size and cutout disposition on amplitude-frequency characteristics is analyzed.

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