**Proceedings of the 3rd International Conference on Nonlinear Dynamics**
**ND-KhPI2010**
**September 21-24, 2010, Kharkov, Ukraine**

# ON SOME CHAOTIC MAPPINGS IN SYMBOL SPACE

**Inese Bula**[1]
University of Latvia and Institute of Mathematics and Computer Science of University of Latvia,
Riga, Latvia

ABSTRACT

A chaotic dynamical system consists of a compact metric space together with a chaotic continuous mapping. A well known chaotic mapping in symbol space is a shift mapping. However, other chaotic mappings in symbol space exist too. We consider the increasing mapping as a generalization of the shift mapping and the *k*-switch mapping too, and combination of both mappings. All these mappings are chaotic. Models with chaotic mappings are not predictable in long-term.

## INTRODUCTION

Chaotic dynamical systems have received a great deal of attention in the past. Chaotic systems are nonlinear dynamical systems with certain distinct characteristics. In this paper we consider chaotic mappings in symbol space.

The technique of characterizing the orbit structure of a dynamical system via infinite sequences of symbols is known as symbolic dynamics. If the process is a discrete process such as the iteration of a function *f*, then the theory hopes to understand the eventual behavior of the points (the orbit of *x* by *f*) $x, f(x), f^2(x),..., f^n(x),...$ as $n$ becomes large. That is, dynamical systems ask to somewhat nonmathematical sounding question: where do points go and what do they do when they get there?

A well known chaotic mapping in symbol space is a shift mapping ([8], [9], [10], [14]). However, other chaotic mappings in symbol space exist too. The basic change is to consider the process (physical or social phenomenon) not only at a set of times which are equally spaced, say at unit time apart (a shift mapping), but at a set of times which are not equally spaced, say if we cannot fixed unit time (an increasing mapping).There is a philosophy of modeling in which we study idealized systems that have properties that can be closely approximated by physical systems. The experimentalist takes the view that only quantities that can be measured have meaning. This is a mathematical reality that underlies what the experimentalist can see.

The paper is structured as follows. It starts with preliminaries concerning notations and terminology that is used in the paper followed by a definition of the chaotic mapping. The increasing mapping and the *k*-switch mapping is considered in Section 2. The combination of both listed mappings is considered in Section 3. Finally we give some conclusions.

## 1. PRELIMINARIES

The section presents the notation and terminology used in this paper. Terminology comes from combinatorics on words (for example, [12] or [13]).

We give some notations at first: $\overline{k,n} = \{k, k+1,..., n\}$, $k \le n$ and $k, n \in \{0,1,2,...\}$,

$Z$ is the set of integers, $Z_+ = \{x | x \in Z \;\&\; x > 0\}$, $N = Z_+ \cup \{0\}$.

From now on *A* will denote a finite *alphabet*, i.e., a finite nonempty set $\{a_0, a_1, a_2,..., a_n\}$ and the elements are called *letters*. We assume that *A* contains at least two symbols. By $A^*$ we will denote the set of all finite sequences of letters, or finite *words*, this set contains empty word (or sequence) $\lambda$ too. $A^+ = A^* \setminus \{\lambda\}$. A word $\omega \in A^+$ can be written uniquely as a sequence of letters

[1] Corresponding author. Email ibula@lanet.lv

as $\omega = \omega_1\omega_2...\omega_l$, with $\omega_i \in A$, $1 \le i \le l$. The integer $l$ is called the *length* of $\omega$ and denoted $|\omega|$. The length of $\lambda$ is 0. An extension of the concept of finite word is obtained by considering infinite sequences of symbols over a finite set. *One-sided* (from left to right) infinite sequence or word, or simply *infinite word*, over $A$ is any total map $\omega{:}N \to A$. The set $A^\omega$ contains all infinite words. $A^\infty = A^* \cup A^\omega$. If the word $u = u_0u_1u_2... \in A^\infty$, where $u_0, u_1, u_2,... \in A$, then finite word $u_0u_1u_2...u_n$ is called the *prefix* of *u* of length *n+1*. The empty word $\lambda$ is assumed to be the prefix of *u* of length 0. $\mathrm{Pref}(u) = \{\lambda, u_0, u_0u_1, u_0u_1u_2,..., u_0u_1u_2...u_n,...\}$ is the set of all prefixes of word *u*.

Secondly we introduce in $A^\infty$ a metric *d* as follows.

**Definition 1.1** ([13]). Let $u, v \in A^\infty$. The mapping $d: A^\infty \times A^\infty \to R$ is called a *metric* (*prefix metric)* in the set $A^\infty$ if $d(u,v) = \begin{cases} 2^{-m}, u \ne v, \\ 0, \quad u = v, \end{cases}$ where $m = \max\{|\omega| \,|\, \omega \in \mathrm{Pref}(u) \cap \mathrm{Pref}(v)\}$.

The metric space $(A^\omega, d)$ is compact metric space ([15]).

The term “chaos” in reference to functions was first used in Li and Yorke's paper “Period three implies chaos” ([11], 1975). Although there is no universally accepted mathematical definition of chaos, Devaney’s definition [6] of chaos is one of the most popular. In order to introduce the definition of chaos in the sense of Devaney, we first present several preliminary concepts.

Let $(X, \rho)$ be metric space and $A \subset X$ and $F \subset A$.

We say that the set *F* is *dense* in *A* ([6], [8], [14]) if for each point *x* in *A* and each $\varepsilon > 0$, there exists *y* in *F* such that $\rho(x, y) < \varepsilon$.

We say that the function *f* is *topologically transitive* on *A* ([6], [8], [14]) if for any two points *x* and *y* in *A* and any $\varepsilon > 0$, there is $z \in A$ such that $\rho(z, x) < \varepsilon$ and $\rho(f^n(z), y) < \varepsilon$ for some *n*.

We say that the function $f: A \to A$ exhibits *sensitive dependence on initial conditions* ([6], [8], [14]) if there exists a $\delta > 0$ such that for any *x* in *A* and any $\varepsilon > 0$, there is a *y* in *A* and natural number *n* such that $\rho(x, y) < \varepsilon$ and $\rho(f^n(x), f^n(y)) > \delta$.

**Definition 1.2** ([6]). The function $f: A \to A$ is *chaotic* if

a) the set of periodic points of *f* are dense in *X*,
b) *f* is topologically transitive and
c) *f* exhibits sensitive dependence on initial conditions.

Devaney's definition is not the unique classification of a chaotic map. For example, another definition can be found in [14]. Also mappings with only one property - sensitive dependence on initial conditions - frequently are considered as chaotic (see [7]). Banks, Brooks, Cairns, Davis and Stacey [1] have demonstrated that for continuous functions, the defining characteristics of chaos are topological transitivity and the density of the set of periodic points. But if the set of periodic points of function *f* is dense in *A* and there is a point whose orbit under iterations of *f* is dense in the set *A*, then *f* is topologically transitive on *A* ([8]). Therefore in this case if *f* is invariant in the set *A* and continuous, then it is chaotic mapping.

## 2. INCREASING MAPPING AND *k*-SWITCH MAPPING

We have introduced the notion of increasing mapping in [3].

Let $f_\omega(x) = x_{g(0)}x_{g(1)}x_{g(2)}...x_{g(i)}...,\ i \in N,\ x \in A^\omega$. In this case the function *g* is called *the generator function* of mapping $f_\omega$.

**Definition 2.1** ([3]). A function $g: N \to N$ is called *positively increasing function* if $0 < g(0)$ and $\forall i\, \forall j\, [i < j \Rightarrow g(i) < g(j)]$. The mapping $f_\omega: A^\omega \to A^\omega$ is called *increasing mapping* if its generator function $g: N \to N$ is positively increasing.

The well known shift map is increasing mapping in one-sided infinite symbol space $A^\omega$, in this case the generator function is a positively increasing function $g: N \to N$, where $g(x) = x + 1$.

**Theorem 2.1** ([3]). *The increasing mapping* $f_\omega : A^\omega \to A^\omega$ *is continuous in the set* $A^\omega$*.For increasing mapping* $f_\omega : A^\omega \to A^\omega$ *exists a dense orbit in the set* $A^\omega$*. The set of periodic points set of increasing mapping* $f_\omega : A^\omega \to A^\omega$ *is dense in the set* $A^\omega$*.*

**Theorem 2.2** ([3]). *The increasing mapping* $f_\omega : A^\omega \to A^\omega$ *is chaotic in the set* $A^\omega$*.*

Now we can give two **conclusions** about mappings in symbol space which are not chaotic:

1) If the generator function $g : N \to N$ of mapping $f_\omega : A^\omega \to A^\omega$ is such that $g(0) = 0$, then the generated mapping $f_\omega$ is not chaotic in the set $A^\omega$;

2) If the generator function $g : N \to N$ of mapping $f_\omega : A^\omega \to A^\omega$ is not one-to-one function, then the generated mapping $f_\omega$ is not chaotic in the set $A^\omega$.

We have introduced the notion of *k*-switch mapping in [4].

**Definition 2.2.** The mapping $f_{\overline{1k}} : A^\omega \to A^\omega$ is called *k-switch* $(k \in N)$ *mapping* if for every $s = s_0 s_1 s_2 s_3 \ldots s_k s_{k+1} \ldots \in A^\omega$: $f_{\overline{1k}}(s) = \overline{s_1 s_2 s_3} \ldots \overline{s_k} s_{k+1} s_{k+2} \ldots$,
where $\overline{s_i}, i = \overline{1,k}$, there is the same symbol (letter) as $s_i$ or $\exists a \in A : \overline{s_i} = a$. In other words, at first, this mapping is shift and, secondly, this mapping switches some symbols (not more as *k* symbols).

For example, let $A = \{0,1\}$ and $f_{1,3}(s_0 s_1 s_2 s_3 s_4 \ldots) = \overline{s_1} s_2 \overline{s_3} s_4 s_5 \ldots$ is 3-switch mapping that switch first and third symbol. Indices by mapping *f* show which symbols switches to another by defined rule. For example, if $s = 1111111\ldots$, then $f_{1,3}(s) = 0101111\ldots$. If we consider situation with $A$ that contains at least three symbols $A = \{a_0, a_1, a_2, \ldots, a_n\}$, then we define rule which $a_i$ switches to $a_j$ and for every $a_j \in A$ only one $a_i \in A$ exists with this rule.

More formally: we set a bijection $\overline{\phantom{a}} : A \to A$, we fix indices $1 \le i_1 < i_2 < \ldots < i_n = k$. Then

$$f_{\overline{1k}}(s_0 s_1 s_2 \ldots) = t_0 t_1 t_2 \ldots,$$

$$\text{where } t_j = \begin{cases} \overline{s_{j+1}}, \text{ if } \exists \eta \; i_\eta = j+1, \\ s_{j+1}, \text{ otherwise.} \end{cases}$$

**Theorem 2.3** ([4]). *The k-switch mapping* $f_{\overline{1k}} : A^\omega \to A^\omega$ *is continuous in the set* $A^\omega$*.The k-switch mapping* $f_{\overline{1k}} : A^\omega \to A^\omega$ *is topologically transitive in the set* $A^\omega$*. The set of periodic points of k-switch mapping* $f_{\overline{1k}} : A^\omega \to A^\omega$ *is dense in the set* $A^\omega$*.*

**Theorem 2.4** ([4]). *The k-switch mapping* $f_{\overline{1k}} : A^\omega \to A^\omega$ *is chaotic in the set* $A^\omega$*.*

We have demonstrated two different classes of chaotic mappings. It is possible for increasing mapping (from two symbols 0 and 1 space) to construct corresponding mapping in the unit segment that is chaotic ([2], [5]). Similarly we can obtain the chaotic map in the interval [0, 1] from every chaotic mapping of two symbols 0 and 1 space.

## 3. COMBINATION OF INCREASING MAPPING AND *k*-SWITCH MAPPING

Now we consider the new class of mappings.

**Definition 3.1.** The mapping $f_{\alpha|\beta} : A^\omega \to A^\omega$ is called *increasing-switch mapping* if for every $s = s_0 s_1 s_2 s_3 \ldots s_k s_{k+1} \ldots \in A^\omega$, firstly, some symbols are “forgets” – these indices of symbols are designated in $\alpha$ part, secondly, some symbols are switched to another's - these indices of symbols are designated in $\beta$ part and thirdly, $\alpha \cap \beta$ is an empty set.

For example, let $A = \{0,1\}$ and

$$f_{0,2,4|1,5}(s_0 s_1 s_2 s_3 s_4 s_5 \ldots) = \overline{s_1} s_3 \overline{s_5} s_6 s_7 \ldots$$

This mapping "forgets" symbols $s_0, s_2, s_4$ and switch symbols $s_1, s_5$. Exactly, if we consider the infinite sequence $s = 0110\,010111000\ldots \in A^{\omega}$, then

$$f_{0,2,4|1,5}(s) = 0000111000\ldots$$

If we consider situation with $A$ that contains more than two symbols $A = \{a_0, a_1, a_2, \ldots, a_n\}$, then we define the rule which $a_i$ switches to $a_j$ and for every $a_j \in A$ exist only one $a_i \in A$ with this rule (analogical as case of $k$-switch mapping).

We note that the increasing-switch mapping is not a function composition of increasing mapping and $k$-switch mapping in general case since $f_{\omega}(f_{\overline{1k}}(s))$ and $f_{\overline{1k}}(f_{\omega}(s))$ "forgets" two first symbols of the sequence $s$. But, on the other hand, every composition of increasing mapping and $k$-switch mapping ( $f_{\omega}(f_{\overline{1k}}(s))$ or $f_{\overline{1k}}(f_{\omega}(s))$) is increasing-switch mapping .

It is possible to show that every mapping that changes the finite number of symbols ("forgets", switch with or without bijection rule, or another changes) is continuous mapping in metric space $(A^{\omega}, d)$. But every continuous mapping in this space is not chaotic (see conclusions below Theorem 2.2).

**Theorem 3.1.** *The increasing-switch mapping* $f_{\alpha|\beta} : A^{\omega} \to A^{\omega}$ *is continuous in the set* $A^{\omega}$*.The increasing-switch mapping* $f_{\alpha|\beta} : A^{\omega} \to A^{\omega}$ *is topologically transitive in the set* $A^{\omega}$. *The set of periodic points of increasing-switch mapping* $f_{\alpha|\beta} : A^{\omega} \to A^{\omega}$ *is dense in the set* $A^{\omega}$.

**Theorem 3.2.** *The increasing-switch mapping* $f_{\alpha|\beta} : A^{\omega} \to A^{\omega}$ *is chaotic in the set* $A^{\omega}$.

By Lind and Marcus [10] terminology: a dynamical system $(X, f)$ consists of a compact metric space $X$ together with a continuous map $f : X \to X$. We have found three dynamical systems $(A^{\omega}, f_{\omega})$, $(A^{\omega}, f_{\overline{1k}})$ and $(A^{\omega}, f_{\alpha|\beta})$ which all are chaotic.

## CONCLUSIONS

Let

$$x(t_0), x(t_1), \ldots, x(t_n), \ldots$$

be the flow of discrete signals. Suppose that we have the experimentally observed subsequence

$$x(T_0), x(T_1), \ldots, x(T_n), \ldots.$$

If

$$T_0 = t_1, T_1 = t_2, \ldots, T_n = t_{n+1}, \ldots,$$

then we have the shift map. Notice if we have the infinite word

$$x = x_0 x_1 \ldots x_n \ldots$$

instead of flow of discrete signals, then we have respectively the infinite word

$$y = y_0 y_1 \ldots y_n \ldots$$

instead of the experimentally observed subsequence. Here $\forall t \quad y_t = x_{t-1}$. Hence, we obtain the shift map $g(t) = t + 1$,namely,

$$y = f_{\omega}(x) = x_{g(0)} x_{g(1)} x_{g(2)} \ldots x_{g(n)} \ldots$$

We do not claim that the function $g(t) = t + 1$ is chaotic on the real line $\boldsymbol{R}$ but we had proved that this function as a generator creates the chaotic map $f_{\omega}$ in the symbol space $A^{\omega}$. We had proved something more, namely, every positively increasing function $g$ as a generator creates the chaotic map $f_\omega$ in the symbol space $A^\omega$. In other words, if we had detected in our experiment only subsequence

$$x(t_1), x(t_3), \ldots, x(t_{2n-1}), \ldots$$

even then we can reveal chaotic behavior.

Now we have proved that every $k$-switch mapping $f_{\overline{1k}} : A^\omega \to A^\omega$ and every increasing-switch mapping $f_{\alpha \mid \beta} : A^\omega \to A^\omega$ are chaotic in the symbol space $A^\omega$. In other words, if we had detected in our experiment only subsequence $x(t_1), x(t_2), \ldots, x(t_n), \ldots$ with some kind of regular distortion, even then we can reveal chaotic behavior.

**REFERENCES**


[1] Banks J., Brooks J., Cairns G., Davis G., Stacey P. On Devaney's definition of chaos, *Amer. Math. Monthly,* V.99, pp.29-39, 1992.

[2] Bula I., Topological semi-conjugacy and chaotic mappings, *Proc. of 6th EUROMECH Nonlinear Dynamics Conference (ENOC 2008)*, 2008, http://lib.physcon.ru/?item=1603

[3] Bula I., Buls J., Rumbeniece I. *Why can we detect the chaos?* Journal of Vibroengineering, Vol.10, pp.468-474, 2008.

[4] Bula I., Buls J., Rumbeniece I. On chaotic mappings in symbol space, *Proc. of the 10th conf. on Dynamical Systems – Theory and Applications,* Vol.2, Lodz, Poland, pp.955-962, 2009.

[5] Bula I, Rumbeniece I. Construction of chaotic dynamical system, *Mathematical Modelling and Analysis*, Vol.15 (1), pp.1-8, 2010.

[6] Devaney R. *An introduction to chaotic dynamical systems.* Benjamin Cummings: Menlo Park, CA, 1986.

[7] Gulick D. *Encounters with chaos.* McGraw-Hill, Inc., 1992.

[8] Holmgren R.A. *A first course in discrete dynamical systems*. Universitext, sec. ed., Springer-Verlag, 1996.

[9] Kitchens B.P. *Symbolic Dynamics. One-sided, two-sided and countable state Markov shifts.* Springer-Verlag, 1998.

[10] Lind D., Marcus B. *An introduction to symbolic dynamics and coding.* Cambridge University Press, Cambridge, 1995.

[11] Li T.Y., Yorke J. A. Period three implies chaos. *Amer. Math. Monthly*, V.82(12), P.985-992, 1975.

[12] Lothaire M. Combinatorics on Words. *Encyclopedia of Mathematics and its Applications,* V.17, Addison-Wesley, Reading, MA, 1983.

[13] de Luca A., Varricchio S. *Finiteness and regularity in semigroups and formal languages.* Monographs in Theoretical Computer Science, Springer-Verlag, 1999.

[14] Robinson C. *Dynamical systems. Stability, symbolic dynamics, and chaos.* CRS Press, Boca Raton, Florida, 1995.

[15] Wiggins S. *Global Bifurcations and Chaos. Analytical Methods.* Springer-Verlag, New York, 1988.