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ON THE OSCILLATIONS OF NONLINEAR MAGNETOELASTIC SOLIDS

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ABSTRACT

We investigate the nonlinear oscillations of nonlinear elastic bodies made of magnetoreological materials, i.e. so-called magnetoelastomers. For this materials the magnetic field can be significantly change the material properties, for example, the stiffness parameters. As an example the oscillations of a magnetoelastic sphere is considered. It is shown that the various regimes of oscillation exist.

INTRODUCTION

Magneto-sensitive (MS) or magneto-rheological (MR) elastomers are smart materials whose mechanical properties change significantly under the influence of a magnetic field. They are widely used in the modern engineering as elements of micro-electro-mechanical systems (MEMS) is the integration of mechanical elements, sensors, actuators, and electronics on a common silicon substrate through microfabrication technology, for example, in medical devices. The behavior of MS elastomers under a time-dependent magnetic field is a complex process and up to now not investigated in all details.

MS elastomers are composed of polarizable particles, dispersed in a polymer medium, having the size of the order of few microns (typically from 10^{-7} to 10^{-5} m). Carrier fillers are selected based upon their electro-magnetic and thermo-mechanical properties: silicone and/or other rubber-like materials with a very small electric conductivity. The typical particle volume fraction is between 0.1 and 0.5. During the manufacturing process of MS elastomers, the isotropy condition inherent of the filler material is maintained in the final composite. Therefore, these materials are considered to be isotropic and non-conductive. However, MS elastomers become non-homogeneous due to the presence and distribution of particles in the carrier filler.

Here we formulate an initial-boundary-value problem of a MS elastomer and demonstrate the special features of the dynamic behavior of such system. As an example the nonlinear oscillations of a MS elastic sphere and ring are considered. The basic equations of MS elastomers consist of the equations of motion of the finite elasticity and the Maxwell's field equations for the vector of the magnetic induction. The constitutive equation of MS elastomers described by the strain energy function depending on 6 invariants of the left-Cauchy-Green strain tensor $\mathbf{b}$ and the vector of magnetic induction $\mathbf{B}$, is presented in general. For the sake of simplicity we use the simplified version of the constitutive equation, where the elastomer is assumed to be incompressible and the dependence on the vector of magnetic induction is reduced to the dependence of its magnitude, i.e. the dependence of the strain energy on the mixed invariant is not taken into account.

As an example two one-dimensional problems are considered. The first one is the radial-symmetric deformation of a hollow sphere loaded by external pressure. Using the incompressibility equations the boundary-value problem is reduced to a nonlinear non-autonomous ordinary differential equation of second order with respect to the radial displacement. The magnetic field $\mathbf{B}$ is assumed to be a given periodic function of time. The phase portrait of this equation is obtained. The trajectories can demonstrate the complex behavior. The influence of the material parameters on the solution

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behavior is analyzed in details. For one case of material parameters one can see the weak influence on the oscillations. Other values demonstrate more complex behavior for small frequency with some type of instabilities. The increase of the frequency of the magnetic field $\mathbf{B}$ leads to the stabilization of oscillations near the solution with constant $\mathbf{B}$. It means that using the external magnetic field we can "control" in some sense the motion of the sphere.

1. BASIC EQUATIONS OF INCOMPRESSIBLE MAGNETO-ELASTOMERS

Following [1-4], let us recall the basic relations of the theory of finite magneto-elasticity. For definiteness we consider an incompressible material in the absence of external body forces. The motion of the body is described by the position-vector in the actual configuration $\mathbf{x}$

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{1}$$

while $\mathbf{X}$ is the position-vector in the reference configuration. We use standard notations

$$\mathbf{F} = (\nabla_{\mathbf{X}} \mathbf{x})^{\mathrm{T}}, \quad J = \det \mathbf{F}$$

where $\mathbf{F}$ is the gradient of the position-vector $\mathbf{x}$, $\nabla_{\mathbf{X}}$ is the nabla operator with respect to $\mathbf{X}$. For rubber-like materials we apply the incompressibility condition

$$J = 1 \tag{2}$$

The constitutive equations of an incompressible isotropic magneto-elastic solid are given by

$$\begin{aligned} W &= W(I_1, I_2, I_4, I_5, I_6) \\ \boldsymbol{\sigma} &= -p\mathbf{I} + \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{F}} \\ \mathbf{M}_{\mathrm{e}} &= -\frac{\partial W}{\partial \mathbf{B}} \end{aligned} \tag{3}$$

where W is the specific free energy given composed of the following set of invariants

$$I_1 = \mathrm{tr}\mathbf{b}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\mathbf{b})^2 - \mathrm{tr}\mathbf{b}^2\right], \quad I_4 = B^2 \equiv |\mathbf{B}|^2, \quad I_5 = (\mathbf{b}\cdot\mathbf{B})\cdot\mathbf{B}, \quad I_6 = (\mathbf{b}^2\cdot\mathbf{B})\cdot\mathbf{B}$$

where $\mathbf{b} = \mathbf{F}\cdot\mathbf{F}^T$ is the left-Cauchy-Green strain tensor, $\mathbf{B}$ is the vector of the magnetic induction, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{M}_{\mathrm{e}}$ is the normalized vector of magnetization, p is a Lagrange multiplier associated with the constraint (2), and $\mathbf{I}$ is the second-order identity tensor. From the physical point of view p is the hydrostatic pressure [9, 10].

The equation of motion and the field equation have the following form

$$\begin{aligned} &\nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} + (\nabla_{\mathbf{x}} \mathbf{B}) \cdot \mathbf{M}_{\mathrm{e}} = \rho \dot{\mathbf{v}} \\ &\nabla_{\mathbf{x}} \cdot \mathbf{B} = 0 \end{aligned} \tag{4}$$

where $\nabla_{\mathbf{x}}$ is the nabla operator in the actual configuration, ρ is the density, $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity, $\dot{(\ldots)}$ is the material derivative with respect to the time t. Further we assume that $\mathbf{B}$ is homogeneous and depends only on t. Then Eqs (4) reduce to the standard one

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} = \rho \dot{\mathbf{v}} \tag{5}$$

The static boundary conditions have the standard form

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f} \tag{6}$$

where $\mathbf{n}$ is an outer unit normal to the boundary of the body and $\mathbf{f}$ is an external surface load.

Let us specify the form of the energy given by

$$W = \frac{1}{2}\mu(I_4)(I_1 - 3) \tag{7}$$

with $\mu(I_4) = \mu_0(1 + \eta I_4)$, $\eta > 0$. For small deformations μ_0 is the shear modulus in the absence of the magnetic field, η describes the influence of the magnetic field on the shear modulus. Equation (7) is the classical neo-Hookean model, which is widely used in the mechanics of elastomers (see, for example, [9-11]) with an elastic modulus highly depending on the magnetic field induction intensity. More general constitutive equations were considered, for instance, in [1-8]. Using (7) we obtain

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu(I_4)\mathbf{b} \tag{8}$$

Thus, the boundary-value problem (5), (6) describes the deformations of MS elastomers under action of both the external forces and the magnetic field. Let us note that Eqs (5), (6) contain $\mathbf{B}(t)$ only as a parameter. On the other hand, the dependence $\mathbf{B}$ on t posses one to generate and control the vibrations of MS elastomer based devices. To illustrate this idea we consider an one-dimensional problem for MS elastomers in the next section.

2. NONLINEAR OSCILLATIONS OF A MAGNETO-ELASTIC SPHERE

Following [12] let us consider the oscillations of a hollow magneto-elastic sphere under action of a homogenous magnetic field $\mathbf{B} = \mathbf{B}(t) = B(t)\mathbf{e}_r$ and a inner hydrostatic pressure $\tilde{p}$. In the reference configuration the sphere has the inner and the outer radii r_0 and r_1, respectively. In the spherically symmetric case the position-vector is given by

$$\mathbf{x} = R(r,t)\mathbf{e}_r \tag{9}$$

where $r \in [r_0, r_1]$ is the radial component of the spherical Lagrangian coordinates and $\mathbf{e}_r$ is the appropriate base vector (see, e.g. [9]), R is an unknown function. In the actual configuration the inner and the outer radii are $R_0 = R(r_0,t)$ and $R_1 = R(r_1,t)$, respectively.

From (2) we immediately find that

$$R(r,t) = (r^3 + x(t))^{1/3} \tag{10}$$

where $x(t)$ is a new unknown function. Thus, $R_1 = (r_1^3 + x(t))^{1/3}$, $R_0 = (r_0^3 + x(t))^{1/3}$. Equation (10) is one of the well-known so-called universal solutions for incompressible solids (see, e.g. [9]. For magneto-elastomers the universal solutions are studied in [7]. From (10) it follows that the volume of the sphere is constant, i.e.

$$R_1^3 - R_0^3 = r_1^3 - r_0^3 \tag{11}$$

For the universal solution (10) Eq. (5) is satisfied identically by choosing of $p = p(r,t)$.

For spherically symmetric deformations the boundary conditions are given by

$$\sigma_{RR}(R_1,t) = 0, \quad \sigma_{RR}(R_0,t) = -\tilde{p} \tag{12}$$

where $\sigma_{RR} = \mathbf{e}_r \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_r$. For brevity, we omitted the awkward computations, see, for details, [9], p. 348.

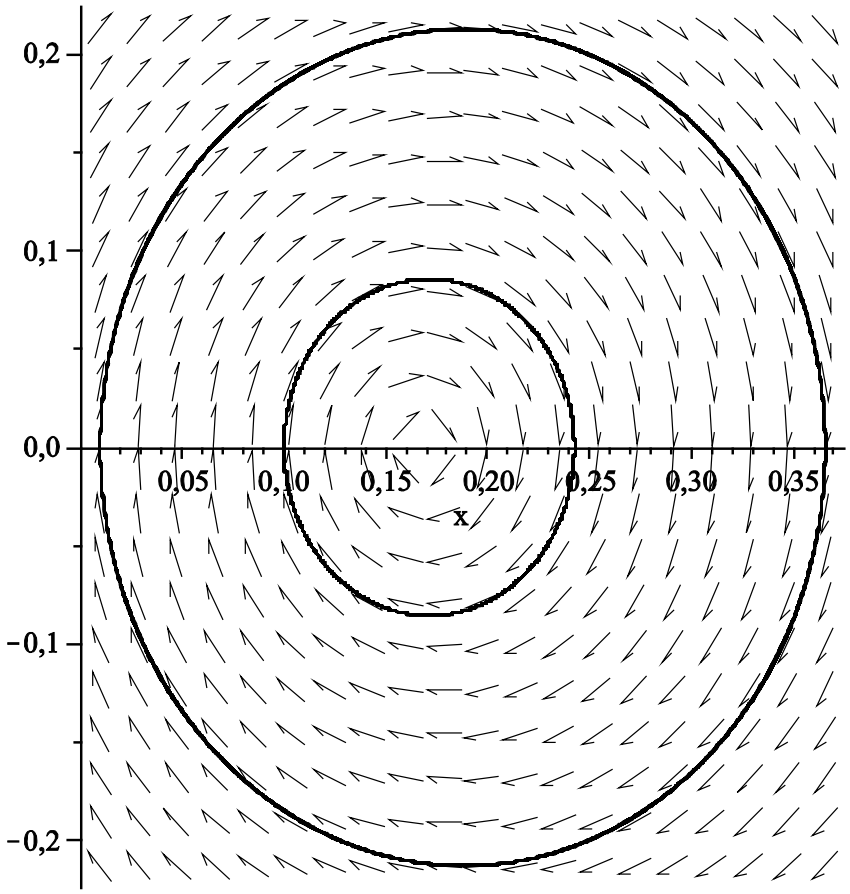

Fig. 1 Phase portrait of (14) in the absence of the magnetic field

Finally, Eqs (5), (12) can be reduced to the ordinary differential equation (ODE) with respect to $x(t)$

$$\rho\left[\ddot{x}\frac{R_1-R_0}{R_1R_0}-\frac{1}{6}\dot{x}^2\frac{R_1^4-R_0^4}{R_1^4R_0^4}\right]+12x\int_{r_0}^{r_1}\frac{R^3+r^3}{R^7}\left(\frac{\partial W}{\partial I_1}+\frac{R^2}{r^2}\frac{\partial W}{\partial I_2}\right)\mathrm{d}r=3\tilde{p} \tag{13}$$

Using (7) Eq. (13) is reduced to

$$\alpha(x)\ddot{x}-\beta(x)\dot{x}^2+\mu\gamma(x)x=\tilde{p} \tag{14}$$

where

$$\alpha(x)=\frac{1}{3}\rho\frac{R_1-R_0}{R_1R_0}>0\,,\ \beta(x)=\frac{1}{18}\rho\frac{R_1^4-R_0^4}{R_1^4R_0^4}>0\,,\ \gamma(x)=4\int_{r_0}^{r_1}\frac{x+2r^3}{\left(x+r^3\right)^{7/3}}\mathrm{d}r$$

are essentially nonlinear functions, and $\gamma(x)$ can not be expressed in elementary functions.

Equation (14) is nonlinear non-autonomous ODE with respect to $x(t)$ which can be solved only numerically. Let us assume a sinusoidal behavior of B: $B(t)=B_0\sin\omega t$, where B_0 is the magnitude, while ω is the frequency.

Examples of numerical simulations are presented in Figs 1, 2. In Fig. 1 the phase portrait of (14) in the absence of the magnetic field ($\mathbf{B}=0$) is shown. Here the following dimensionless parameters $\bar{r}_0=0.9$, $\bar{r}_1=1$ ($\bar{r}=r/r_1$), $\bar{p}\equiv\tilde{p}/\mu=0.01$ are used, and we keep notation x for a new dimensionless variable x/r_1^3. Two closed trajectories correspond to the initial data $x(0)=0.01$, $\dot{x}(0)=0$, and $x(0)=0.1$, $\dot{x}(0)=0$, respectively. In Fig. 2 two trajectories correspond to initial data

$x(0) = 0.01$, $\dot{x}(0) = 0$, and $x(0) = 0.1$, $\dot{x}(0) = 0$, respectively. Here $\overline{\eta} = \eta B_0^2$ and $\overline{\omega} = \omega T$, $T = \sqrt{\rho/\mu_0}$, and the time interval is $[0,300T]$. The trajectories of (14) can demonstrate complex behavior. For the case of low values of $\overline{\eta}$ describing the dependence of the shear modulus on B one can see the weak influence on the oscillations (see top row in Fig. 2). In this case we have the behavior similar to Fig. 1. The middle and bottom rows in Fig. 2 demonstrate more complex behavior for small frequencies, one can see some type of instabilities. The increase of the frequency of B leads to the stabilization of oscillations near the solutions with constant **B** similar to the behavior shown in Fig. 1. It means that using the external magnetic field we can "control" in some sense the motion of the sphere.

The MS cylinder demonstrates the analogous behaviour.

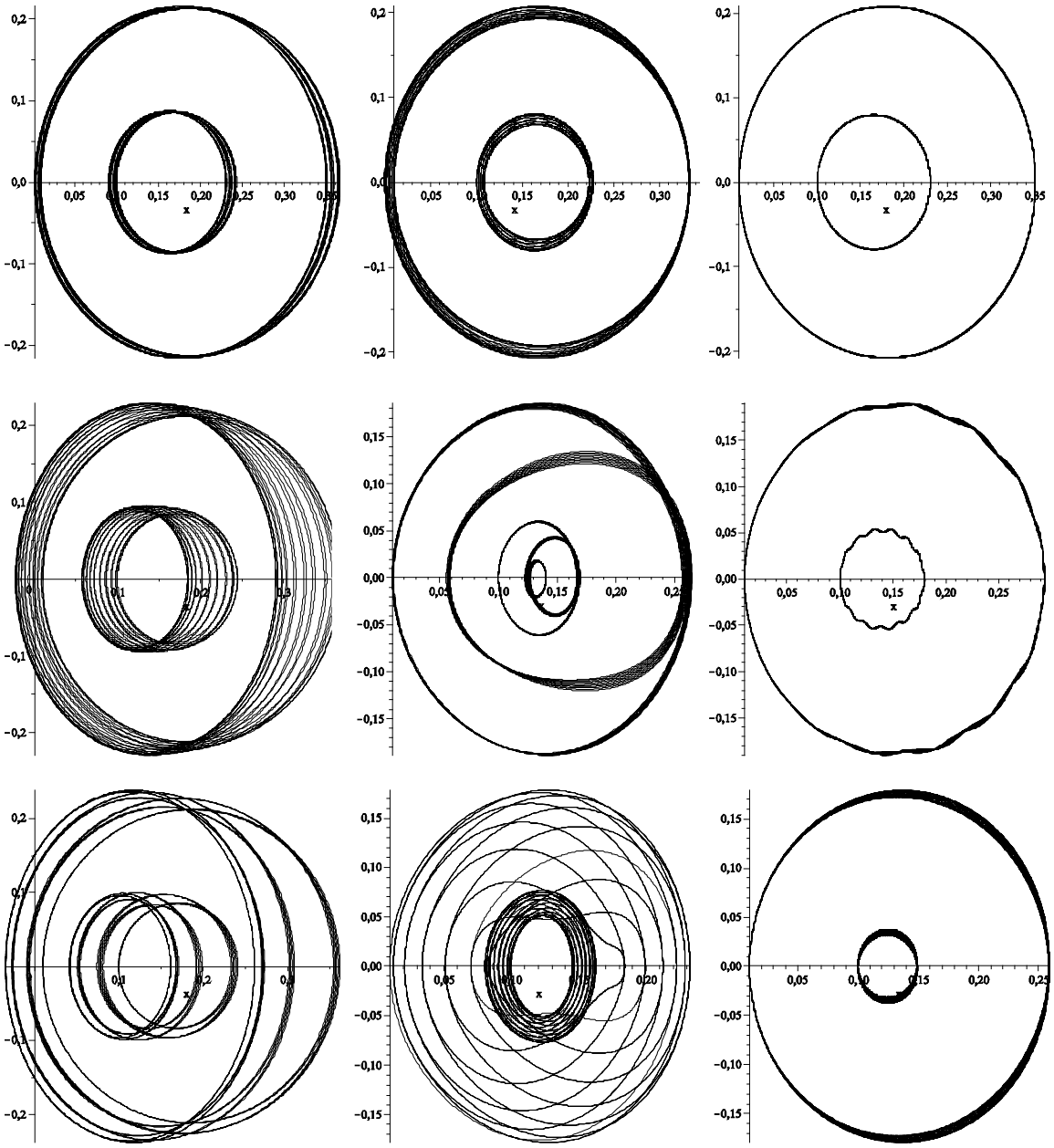

Fig. 2 Examples of trajectories of (14) for different parameters $\overline{\eta}$ and $\overline{\omega}$

CONCLUSIONS

The dynamic statement of the boundary-value problems of MS elastomers under homogeneous with respect to space but time-dependent magnetic field is given. The special property of the boundary-value problem is that the coefficients of the equations of motion may depend on time. As an example, we considered the radially symmetric oscillations of a MS incompressible elastic sphere. It

was shown that using the external magnetic field one can generate and control the oscillations of the sphere. On the other hand, the considered system demonstrates complex behavior which highly depends on the type of external excitation. Such MS elastic sphere under internal pressure may be used, for example, as an actuator or working element of a microengine, based on MS elastomers.

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