**Proceedings of the 3rd International Conference on Nonlinear Dynamics**
**ND-KhPI2010**
**September 21-24, 2010, Kharkov, Ukraine**

# RESEARCH OF NATURAL OSCILLATIONS OF A PLATE WITH MIXED CONDITIONS OF CONTOUR FASTENING

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ABSTRACT

On the basis of the method of perturbation of boundary conditions, the asymptotic method of calculation and analysis of natural oscillations of elastic rectangular plates is proposed. Mixed conditions of the plates fastening of contour, of the "pinched – hinge" type for symmetrical and asymmetrical placement of segments of the fastening, are considered.

## INTRODUCTION

In many cases the application of plates in real constructions is connected with necessity of use of different variants of fastening of certain segments of the plates contours [1]. Dynamic calculation of such constructions makes it necessary to analyze the mathematical models based on boundary value problems with such mixed boundary conditions. The development of methods for constructing solutions of these problems was the subject of the works [2–9]. Systematic work on specified issues suggests that the development of approximate analytical and numerical methods for analyze of mixed boundary value problems of the theory of plates is very topical. At present there are effective different methods of the perturbation theory to solve boundary value problems in the plates and shells theory [3, 8]. Therefore, in this paper, on the base of the method of disturbance of boundary conditions there is proposed the asymptotic method of calculation and analyze of natural oscillations of thin elastic rectangular plates with mixed conditions of the contour fastening of "pinched-hinge" type for symmetric and asymmetric placement of segments of fastening.

## 1. STATEMENT OF THE RESEARCH PROBLEM

One considers the natural oscillations of a rectangular plate with mixed conditions of contour fastening with the help of the method of disturbance of a kind of boundary conditions. The diagram of placement of segments for symmetric *a*) and asymmetric *b*) fastening of a plate is presented in the Figure 1. The appropriate dimensionless differential equations are the following [3]

$$\nabla^4 \overline{w} + \frac{\overline{m}\, b^4}{D} \overline{w}_{tt} = 0, \tag{1}$$

$$\nabla^4 \overline{w} = \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right)^2,$$

where D is the cylindrical rigidity; $\overline{m}$ is the reduced weight of a plate, $a$, $b$ are the sizes of a plate; $\xi$, $\eta$ are the dimensionless coordinates; $\xi = x/b$, $\eta = y/b$ .

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The boundary value problem for the equation (1) is determined by the following mixed boundary conditions [8]

$$w = 0, \; w_{\xi\xi} = 0, \quad k \quad \xi = \pm 0.5 \;, \tag{2}$$

$$w = 0, \; w_{\eta\eta} = \overline{H}(\xi)(w_{\eta\eta} \pm w_{\eta}), \qquad \eta = \pm 0.5, \tag{3}$$

where $k = \dfrac{a}{b}$; $\overline{H}(\xi) = H(\xi - ku) + H(-\xi - ku)$; $H(\xi)$ is the Heaviside function.

## 2. THE METHOD OF CONSTRUCTION OF SOLUTION

For construction of the equation (1) solution, the method of separation of variables can be used:

$$\overline{w} = w(\xi, \eta) \cdot T(t). \tag{4}$$

After substitution of the expression (4) to the equation (1) we can receive two equations:

$$\frac{\partial^2 T}{\partial t^2} + \theta^2 T = 0, \tag{5}$$

$$\nabla^4 w - \lambda w = 0, \tag{6}$$

where $\theta^2$ is circular frequency of natural lateral oscillations of a plate; $\lambda = m\theta^2 b^4/D$ is the eigenvalue of the problem.

*a)* *b)*

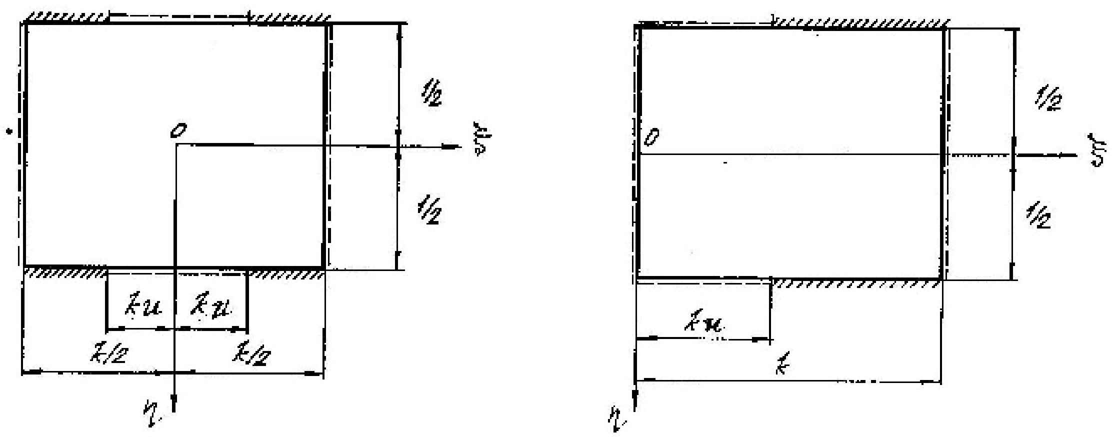


*Fig. 1 The diagram of mixed conditions of fastening of a plate*

According to [8] we enter the parameter ε into boundary conditions (3), under ε = 0 the boundary conditions of the pin-joint type on all boundaries $\eta = \pm 0{,}5$ were realized, and under ε = 1 the initial mixed boundary conditions of the pinched-hinge types are realized:

$$w = 0, \; w_{\eta\eta} = \overline{H}(\xi)\,\varepsilon\,(w_{\eta\eta} \pm w_{\eta}) \;\; \text{under} \;\; \eta = \pm 0{,}5 \,. \tag{7}$$

Under middle values of the parameter ε the mixed boundary conditions of elastic fastening - hinge type with the coefficient of elastic displacement μ = ε/(1–ε) is realized.

For this purpose the small parameter method can be used for the problem natural value λ and the natural form *w*. The following power series by ε are presented:

$$w = \sum_{i=0}^{\infty} w_i \varepsilon^i; \; \lambda = \sum_{i=0}^{\infty} \lambda_i \varepsilon^i \,. \tag{8}$$

Substituting series (8) to the equation (6) and boundary conditions (2), (7) and equating expressions at identical degrees of the parameter ε we are received the following recurrent sequence of the boundary value problems:

$$\varepsilon^0 : \quad \nabla^4 w_0 - \lambda_0 \, w_0 = 0 \,, \tag{9}$$

$$w_0 = 0,\ w_{0\xi\xi} = 0 \quad \text{under} \quad \xi = \pm 0{,}5\,k, \qquad w_0 = 0,\ w_{0\eta\eta} = 0 \quad \text{under} \quad \eta = \pm 0{,}5;$$

$$\varepsilon^j: \quad \nabla^4 w_j - \lambda_0\, w_j = \sum_{i=1}^{j} \lambda_i w_{j-1}, \tag{10}$$

$$w_j = 0,\ w_{j\xi\xi} = 0 \quad \text{under} \quad \xi = \pm 0{,}5k, \qquad w_j = 0,\ w_{j\eta\eta} = \pm \overline{H}(\xi) \sum_{i=0}^{j-1} w_{i\eta} \quad \text{under} \quad \eta = \pm 0{,}5;$$

Let's consider the construction of solution of the problem for a case of directly symmetrical concerning axes ξ and η of the forms of natural oscillations.

In zero approximation one has:

$$w_0 = \xi_0 \cdot \eta_0 = \cos(\frac{\pi m}{k} \cdot \xi) \cdot \cos(\pi n \eta), \tag{11}$$

$$\lambda_0 = \pi^4 \cdot (n^2 + \frac{m^2}{k^2})^2, \quad \text{(n, m) = 1, 3, 5,...} \tag{12}$$

As the first approximation we receive the following boundary value problem:

$$\nabla^4 \cdot w_1 - \lambda_0 \cdot w_1 = \lambda_1 \cdot w_0, \tag{13}$$

$$w_1 = 0, \quad w_{1\xi\xi} = 0 \quad \text{under} \quad \xi = \pm 0{,}5k, \tag{14}$$

$$w_j = 0, \quad w_{j\eta\eta} = \pi\, n (-1)^{\frac{n-1}{2}}\, \overline{H}(\xi) \cos(\frac{\pi\, m}{k} \xi) \quad \text{under} \quad \eta = \pm 0{,}5\,. \tag{15}$$

We can find the solution with the help of the method of decomposition:

$$w_1 = \sum_{i=1,3,...}^{\infty} Y_{1i} \cos\left(\frac{\pi\, i}{k} \xi\right). \tag{16}$$

After the substitution of the given expression to the boundary value problem (13) – (15) two boundary value problems are received:

i = m

$$Y_{1m}^{IV} - 2\frac{\pi^2 m^2}{k^2} Y_{1m}^{II} - \pi^4 n^2 \left(2 \cdot \frac{m^2}{k^2} + n^2\right) Y_{1m} = \lambda_1 \cos(\pi\, n \eta) \tag{17}$$

$$Y_{1m} = 0 \quad Y_{1m}^{II} = \pm \pi \cdot n \cdot (-1)^{\frac{n-1}{2}} \cdot \gamma_{mm} \quad \text{under} \quad \eta = \pm 0{,}5\,. \tag{18}$$

$i \neq m$

$$Y_{1i}^{IV} - 2 \cdot \frac{\pi^2 \cdot i^2}{k^2} \cdot Y_{1i}^{II} - \pi^4 \cdot \left( (n^2 + \frac{m^2}{k^2}) - \frac{i^4}{k^4} \right) \cdot Y_{1i} = 0, \tag{19}$$

$$Y_{1i}^{II} = \pm \pi \cdot n \cdot (-1)^{\frac{n-1}{2}} \cdot \gamma_{im}, \quad Y_{1i} = 0 \quad \text{under} \quad \eta = \pm 0{,}5\,. \tag{20}$$

Here,

$$\gamma_{im} = \begin{cases} 2(0{,}5 - u) - \dfrac{1}{\pi\, m} \sin(2\pi\, m u) & \text{under} \quad i = m, \\[2ex] \dfrac{4}{\pi (m^2 - i^2)} \left[ i \sin(\pi\, u\, i) \cdot \cos(\pi u m) - m \sin(\pi\, u\, m) \cdot \cos(\pi\, u\, i) \right] & \text{under} \quad i \neq m, \end{cases}$$

$$\overline{H}(\xi) \cos(\frac{\pi\, m}{k} \xi) = \sum_{i=1,3,5\,...}^{\infty} \gamma_{im} \cos(\frac{\pi\, i}{k} \xi).$$

Taking into account conditions of uniform of asymptotic decomposition from the boundary value problem (17) – (18) we determine the first correction for the natural value $\lambda_1$

$$\lambda_1 = \frac{-Y_{1m}^{II} Y_0^{I}\Big|_{-0,5}^{0,5}}{\int_{-0,5}^{0,5} Y_0^2 d\eta} = 4\pi^2 n^2 \gamma_{mm}. \tag{21}$$

After determination $\lambda_1$ we are received the following expression for $Y_{1m}$

$$Y_{1m} = \frac{n}{\pi\alpha}\gamma_{mm}\left[\frac{(-1)^{\frac{n-1}{2}}}{2ch\frac{\pi}{2}\beta_1} ch(\pi\beta_1\eta) - \eta s \quad (\pi \quad \eta)\right], \quad = \quad , \tag{22}$$

$$\alpha = n^2 + \frac{m^2}{k^2}; \ \beta_1 = \sqrt{2\frac{m^2}{k^2} + n^2}.$$

The solution of the problem (19) – (20) does not give corrections to the natural value, but it introduces additional items to the form of oscillations:

$$Y_{1i} = \frac{n(-1)^{\frac{n-1}{2}}}{2\pi\left(\frac{i^2}{k^2} + n^2\right)}\gamma_{im}\left[\frac{ch(\alpha_{1i}\eta)}{ch\frac{\alpha_{1i}}{2}} - \frac{\begin{Bmatrix} ch(\varphi_{1i}\eta) \\ cos(\beta_{1i}\eta)\end{Bmatrix}}{\begin{Bmatrix} ch\frac{\varphi_{1i}}{2} \\ cos\frac{\beta_{1i}}{2}\end{Bmatrix}}\right], \begin{Bmatrix} i^2 > m^2 + n^2k^2 \\ i^2 < m^2 + n^2k^2 \end{Bmatrix}, \tag{23}$$

$$\alpha_{1i} = \pi\sqrt{\frac{i^2 + m^2}{k^2} + n^2}; \ \beta_{1i} = \pi\sqrt{\frac{m^2 - i^2}{k^2} + n^2}; \ \varphi_{1i} = \pi\sqrt{\frac{i^2 - m^2}{k^2} - n^2}.$$

Summarizing the expressions (21) and (22) and taking into account the decomposition (16), we receive the first correction to the natural form $w_1$ for directly symmetrical oscillations

$$\begin{aligned} w_1 = \frac{m}{\pi\alpha}\Bigg\{&\gamma_{mm}\left[\frac{(-1)^{\frac{n-1}{2}}}{2ch\frac{\pi}{2}\beta_1} ch(\pi\beta_1\eta) - \eta s \quad (\pi \quad \eta)\right] \quad (\frac{\pi m}{k}\xi) + \\ &+(-1)^{\frac{n-1}{2}}\sum_{i=1,3,5...}^{\infty}\gamma_{im}\left[\frac{ch(\alpha_{1i}\eta)}{ch\frac{\alpha_{1i}}{2}} - \frac{\begin{Bmatrix} ch(\varphi_{1i}\eta) \\ cos(\beta_{1i}\eta)\end{Bmatrix}}{\begin{Bmatrix} ch\frac{\varphi_{1i}}{2} \\ cos\frac{\beta_{1i}}{2}\end{Bmatrix}}\right] cos(\frac{\pi m}{k}\xi)\Bigg\}. \end{aligned} \tag{24}$$

Similarly we receive the expression for the second correction to the natural value $\lambda_2$:

$$\lambda_2 = 4\pi^2 n^2 \gamma_{mm}\left\{1-\frac{\gamma_{mm}}{\pi^2\alpha}\left[\frac{\pi\beta_1}{2} th\frac{\pi\beta_1}{2}+\frac{n^2}{\alpha}-\frac{3}{2}\right]\right\}-$$

$$-\frac{2n^2}{\alpha}\sum_{i=1,3,5...}^{\infty}\gamma_{im}^2\left[\alpha_i th\frac{\alpha_i}{2}+\left\{\begin{matrix}-\varphi_{1i}\, th\dfrac{\varphi_{1i}}{2}\\ \beta_{1i}\, tg\dfrac{\beta_{1i}}{2}\end{matrix}\right\}\right]. \tag{25}$$

Considering the central symmetrical forms of oscillations, similarly we receive the analytical expressions of the natural values and forms for any values of wave numbers by the way of sections of series of disturbance:

$$\lambda = \pi^4\alpha^4 + 4\pi^2\alpha^2\gamma_{mm}\varepsilon + \left\{4\pi^2 n^2\gamma_{mm}\left(1-\frac{\gamma_{mm}}{\pi^2\alpha}\left[\frac{\pi\beta_1}{2} cth^{(-1)^m}\frac{\pi\beta_1}{2}+\frac{n^2}{\alpha}-\frac{3}{2}\right]\right)-\right.$$

$$\left.-\frac{2n^4}{\alpha}\sum_{\left\{\begin{matrix}i=1,3,5...\\ i=2,4,6...\end{matrix}\right\}}^{\infty}\gamma_{im}^2\left[\alpha_i\, cth^{(-1)^i}\frac{\alpha_{1i}}{2}+\left\{\begin{matrix}-\varphi_{1i}\, cth^{(-1)^i}\dfrac{\varphi_{1i}}{2}\\ \beta_{1i}\, ctg^{(-1)^i}\dfrac{\beta_{1i}}{2}\end{matrix}\right\}\right]\right\}\varepsilon^2+..., \tag{26}$$

where

$$\gamma_{im} = \begin{cases} 2(0,5-u)+\dfrac{(-1)^m}{\pi m} sin(2\pi m u) & \text{under} \quad i=m, \\ \dfrac{4}{\pi(m^2-i^2)}\left[\left\{\begin{matrix} i\\ m\end{matrix}\right\} sin(\pi u i)\cdot cos(\pi u m)-\left\{\begin{matrix} m\\ i\end{matrix}\right\} sin(\pi u m)\cdot cos(\pi u i)\right] & \text{under} \quad i\neq m. \end{cases}$$

The problem of determination of natural frequencies of a plate having asymmetric segments of mixed boundary conditions (pic.1, b) can be decided the same way.

Further section of the series for the natural value must be rebuilt to the fractional rational function AP and we can calculate the the first natural value of the boundary value problem (6), (7), (2) under $\varepsilon = 1$.

## 3. THE ANALYSIS

Results of numerical analysis of dependence of the natural vibration frequency from the size of the segment are presented in Figure 2.

The solutions obtained on the basis of proposed approach for a plate with symmetric and asymmetric placement of the segment of substitution is represented by the curves 1, 2 and 3. The results obtained by the method of integral equations [9], 4, 5 are experimental data [9] and 6 are the results for some middle values of u, obtained by the dual series [7]. For the limiting case $\varepsilon = 1$, with complete pinch of the boundary of the plate $\eta = \pm\, 0,5$ the first natural value problem computed on the basis of the constructed solution is equal to $\lambda=1,7081\pi^4$, as obtained numerically [5, 6] – $\lambda=1,7050\pi^4$ (error is 0, 18%). Analysis of the data shows that in general, the discrepancy of the results obtained with certain items does not exceed 2%.

For the case of symmetrical placement of the segments of pinch the relation of the natural value to the sizes of the segment hinged pin-joint has three reference segments. On the first segment from u = 0,0 up to u = 0,05 with increase of parameter u the natural value decreases insignificantly. On the following segment from u = 0,05 up to u = 0,4 the natural values decrease almost linearly. On the third segment from u = 0,4 up to u = 0,5 the decreasing of the natural value also is insignificant. For the plate with asymmetrical placement of the segment of pinch these zones will place in the following limits of the parameter u: the first one, u = 0,0-0,2; for the second one, u = 0,2-0,8; for the third one, u = 0,8-1,0.

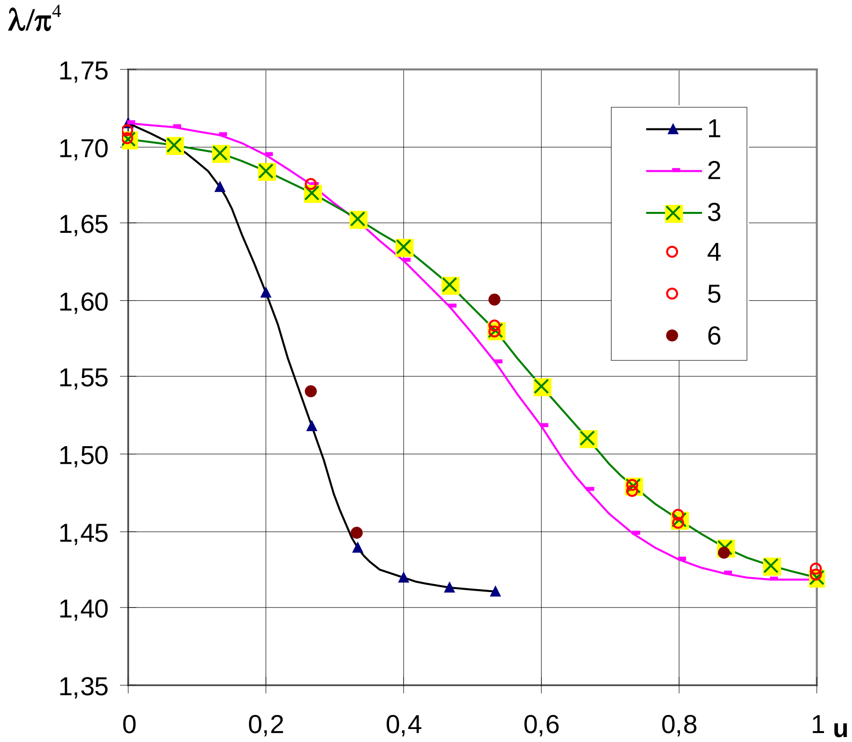


*Fig. 2 The frequency of natural oscillations of pinch of the segment*

## CONCLUSIONS

The given results show that small effect of the small segments of hinged pin-joint type (the first segment of the curve), and small segments of pinching (the third segment of the curve) influence a little on the natural frequency. There are also such ratios of the sizes of segments of mixed boundary conditions, at which the little change can essentially influence on the plate frequency (middle segment of the curve).

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