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SOME REMARKS ABOUT QUASI-STEADY DYNAMICS

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ABSTRACT

A mechanical system consisting of two interacting subsystems is considered. When the interaction is removed, one subsystem is Hamiltonian and the other one is a dissipative linear oscillatory system. Integral manifolds theory is used to study the motions that are established after the high-frequency normal oscillations of the dissipative subsystem are damped. Evolution equations are constructed to describe a behavior of the Hamiltonian subsystem over long time interval.

1. SYSTEM DESCRIPTION. BASIC ASSUMPTIONS

We consider a dynamical system consisting of two interacting subsystems, S_H and S_D. When the interaction is removed, the subsystem S_H becomes the Hamiltonian system with n degrees of freedom and the subsystem S_D becomes a dissipative linear oscillatory system with m degrees of freedom. The characteristic period of oscillations in subsystem S_D and the characteristic damping time of these oscillations are comparable in magnitude and much smaller than the characteristic time of motions in S_H. Below we will call S_H the damped system and S_D the damper one.

The equations of motion of the system $S_H + S_D$ can be written in Routhian form:

$$\mathbf{P}^\bullet = -\nabla_{\mathbf{Q}} R, \quad \mathbf{Q}^\bullet = \nabla_{\mathbf{P}} R, \qquad (\nabla_{\mathbf{v}} R)^\bullet - \nabla_{\mathbf{q}} R = -\nabla_{\mathbf{v}} \Phi \tag{1}$$

Here $\mathbf{P} = (P_1, \ldots, P_n)^T$ and $\mathbf{Q} = (Q_1, \ldots, Q_n)^T$ are canonical variables used to describe the motion in S_H, $\mathbf{q} = (q_1, \ldots, q_m)^T$ is the generalized coordinate vector of the damper with $\mathbf{v} = \mathbf{q}^\bullet$. Dots denote derivatives with respect to time t.

The Routhian function R in (1) is a combination of the Hamiltonian H of subsystem S_H, the Lagrangian L of subsystem S_D, and a function K characterizing the interaction of the subsystems:

$$R = H + K - L$$

Given these assumptions the Lagrangian L and the dissipative function Φ of the damper can be written in the form

$$L(\mathbf{v}, \mathbf{q}, \varepsilon) = \frac{1}{2}\left[(\mathbf{v}, M\mathbf{v}) - \varepsilon^{-2}(\mathbf{q}, \Lambda\mathbf{q})\right], \qquad \Phi(\mathbf{v}, \varepsilon) = \frac{1}{2\varepsilon}(\mathbf{v}, D\mathbf{v}) \tag{2}$$

$$\varepsilon = T_D / T_H \ll 1$$

Here M, Λ and D are positive-definite symmetric matrices with constant coefficients, T_D and T_H are characteristic times of processes in S_D and S_H respectively.

We take

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$$K(\mathbf{P},\mathbf{Q},\mathbf{v},\mathbf{q}) = (\mathbf{u},\mathbf{q}) + \frac{1}{2}(\mathbf{v},\Gamma\mathbf{q}) + K_2(\mathbf{P},\mathbf{Q},\mathbf{q})$$

to be the interaction function with $\mathbf{u} = (u_1(\mathbf{P},\mathbf{Q}),\ldots,u_m(\mathbf{P},\mathbf{Q}))^T = \nabla_{\mathbf{q}} K(\mathbf{P},\mathbf{Q},0,0)$, Γ is an antisymmetric matrix whose elements are functions of $\mathbf{P}$, $\mathbf{Q}$ and the function $K_2(\mathbf{P},\mathbf{Q},\mathbf{q}) = O(q^2)$, $q = |\mathbf{q}| = (q_1^2 + \ldots + q_m^2)^{1/2}$.

With this choice of K the system $S_H + S_D$ is a finite-dimensional model of systems encountered in studies of the motion of a deformable solid about its centre of mass (Section 5).

2. MAIN THEOREM

When studying the dynamics of $S_H + S_D$ over time intervals comparable to or substantially greater than T_H, it is desirable to consider the motion of the damper to be forced and to describe it by the relations of the form

$$\mathbf{v} = \mathbf{v}_*(\mathbf{P},\mathbf{Q},\varepsilon), \quad \mathbf{q} = \mathbf{q}_*(\mathbf{P},\mathbf{Q},\varepsilon) \tag{3}$$

Substituting (3) into the equations for $\mathbf{P}^\bullet$, $\mathbf{Q}^\bullet$ in (1) we obtain a closed system of equations describing the behavior of subsystem S_H after the normal oscillations of the damper have decayed away.

Various modifications of these equations for the quasi-steady motions of specific systems were constructed in [1-3]. There have been attempts [4,5] to give a justification for using such equations to describe the regular components of the motion by boundary function theory methods [6].

Relations (3) define a hypersurface Σ, $\dim\Sigma = 2m$ in the phase space of the system $S_H + S_D$. If this hypersurface is invariant with respect to the phase flow of the system, it is called an integral manifold (IM) [7,8].

Theorem. *For sufficiently small values of the parameter ε system (1) possesses an IM Σ described by the relations of the form (3). On the manifold Σ system (1) is equivalent to the system*

$$\begin{aligned}\mathbf{P}^\bullet &= -\nabla_{\mathbf{Q}} H - \nabla_{\mathbf{Q}} K(\mathbf{P},\mathbf{Q},\mathbf{v}_*(\mathbf{v},\mathbf{q},\varepsilon),\mathbf{q}_*(\mathbf{v},\mathbf{q},\varepsilon)) \\ \mathbf{Q}^\bullet &= \nabla_{\mathbf{P}} H + \nabla_{\mathbf{P}} K(\mathbf{P},\mathbf{Q},\mathbf{v}_*(\mathbf{v},\mathbf{q},\varepsilon),\mathbf{q}_*(\mathbf{v},\mathbf{q},\varepsilon))\end{aligned} \tag{4}$$

The functions $\mathbf{v}_*(\mathbf{P},\mathbf{Q},\varepsilon)$, $\mathbf{q}_*(\mathbf{P},\mathbf{Q},\varepsilon)$ satisfy the inequalities

$$|\mathbf{v}_*(\mathbf{P},\mathbf{Q},\varepsilon)| \le \varepsilon^2 C_1, \quad |\mathbf{q}_*(\mathbf{P},\mathbf{Q},\varepsilon)| \le \varepsilon^2 C_1, \quad C_1 = \text{const} > 0$$

The proof of this theorem consists of constructing a special contraction mapping $\Im$ on the set of functions specifying hypersurfaces in phase space [9].

3. APPROXIMATE EQUATIONS FOR QUASI-STEADY MOTION

It is not difficult to find that in quasi-steady motion

$$\mathbf{v} = -\varepsilon^2 \Lambda^{-1}\{\mathbf{u}, H\} \tag{5}$$

with an error of $O(\varepsilon^3)$, and

$$\mathbf{q} = -\varepsilon^2 \Lambda^{-1}\mathbf{u} + \varepsilon^3 \Lambda^{-1} D \Lambda^{-1}\{\mathbf{u}, H\} \tag{6}$$

with an error of $O(\varepsilon^3)$. Here $\{\cdot,\cdot\}$ are Poisson brackets for the subsystem S_H.

Substituting expressions (5),(6) into (4) we obtain a system of approximate equations for the quasi-steady motion

$$\mathbf{P}^{\bullet} = -\nabla_{\mathbf{Q}}\hat{H} - \varepsilon^3 U_{\mathbf{Q}}\Lambda^{-1}D\Lambda^{-1}\{\mathbf{u},H\} \qquad (7)$$
$$\mathbf{Q}^{\bullet} = \nabla_{\mathbf{P}}\hat{H} + \varepsilon^3 U_{\mathbf{P}}\Lambda^{-1}D\Lambda^{-1}\{\mathbf{u},H\}$$

where

$$\hat{H}(\mathbf{P},\mathbf{Q},\varepsilon) = H(\mathbf{P}.\mathbf{Q}) + \varepsilon^2 H_2(\mathbf{P},\mathbf{Q}), \qquad H_2(\mathbf{P},\mathbf{Q}) = -\frac{1}{2}(\mathbf{u},\Lambda^{-1}\mathbf{u})$$

$$U_{\mathbf{P}} = \begin{Vmatrix} \frac{\partial u_1}{\partial P_1} & \dots & \frac{\partial u_m}{\partial P_1} \\ \dots & \dots & \dots \\ \frac{\partial u_1}{\partial P_n} & \dots & \frac{\partial u_m}{\partial P_n} \end{Vmatrix}, \quad U_{\mathbf{Q}} = \begin{Vmatrix} \frac{\partial u_1}{\partial Q_1} & \dots & \frac{\partial u_m}{\partial Q_1} \\ \dots & \dots & \dots \\ \frac{\partial u_1}{\partial Q_n} & \dots & \frac{\partial u_m}{\partial Q_n} \end{Vmatrix}$$

The nearly-Hamiltonian system of equations (7) describes the influence of the interaction with the damper on the dynamics of subsystem S_H to an accuracy of $O(\varepsilon)$ over a time interval ε^{-3}.

4. EVOLUTION OF QUASI-STEADY MOTION IN AN INTEGRABLE SUBSYSTEM S_H

Suppose that $\mathbf{I} = (I_1,\dots,I_n)^T$, $\varphi = (\varphi_1,\dots,\varphi_n)^T$ are "action-angle" variables in S_H. In $\mathbf{I},\varphi$ variables the equations of quasi-steady motion have the form

$$\mathbf{I}^{\bullet} = -\varepsilon^2\nabla_{\varphi}H_2 - \varepsilon^3 U_{\varphi}\Lambda^{-1}D\Lambda^{-1}U_{\varphi}^T\omega \qquad (8)$$
$$\varphi^{\bullet} = \omega(\mathbf{I}) + \varepsilon^2\nabla_{\mathbf{I}}H_2 + \varepsilon^3 U_{\mathbf{I}}\Lambda^{-1}D\Lambda^{-1}U_{\varphi}^T\omega$$

Here $\omega(\mathbf{I}) = \nabla_{\mathbf{I}}H(\mathbf{I})$ is the frequency vector of the subsystem S_H.

The variables of (8) separate: the $\mathbf{I}$ variables are slow ($\mathbf{I}^{\bullet} = O(\varepsilon^2)$) and the φ variables are fast ($\varphi^{\bullet} = O(1)$).

We shall study the behavior of the slow variables using an averaging method [10]. For simplicity we restrict ourselves to the case when the Fourier series of the function $\mathbf{u}(\mathbf{I},\varphi)$ with respect to φ contains a finite number of terms

$$\mathbf{u}(\mathbf{I},\varphi) = \sum_{\mathbf{k}\in Z^n, |\mathbf{k}|\le N} \mathbf{u}_{\mathbf{k}}(I)e^{i<\mathbf{k},\varphi>}, \qquad <\mathbf{k},\varphi> = k_1\varphi_1 + \dots + k_n\varphi_n$$

In system (8) we perform two consecutive averaging changes of variables

$$(\mathbf{I},\varphi) \xrightarrow{1} (\tilde{\mathbf{I}},\tilde{\varphi}) \xrightarrow{2} (\tilde{\tilde{\mathbf{I}}},\tilde{\tilde{\varphi}})$$

The first change of variables removes the second-order terms in ε in the slow variable equations and is a canonical transformation with generating function

$$S(\tilde{\mathbf{I}},\varphi) = <\tilde{\mathbf{I}},\varphi> - i\varepsilon^2 \sum_{\mathbf{k}\in Z^n\backslash 0, |\mathbf{k}|\le 2N} \frac{H_{2\mathbf{k}}(\tilde{\mathbf{I}})}{<\mathbf{k},\varphi>} e^{i<\mathbf{k},\varphi>}$$

where

$$H_{2\mathbf{k}}(\mathbf{I}) = -\frac{1}{2}\sum_{\mathbf{k}'\in Z^n,\, |\mathbf{k}'|\le N,\, |\mathbf{k}-\mathbf{k}'|\le N} (\mathbf{u}_{\mathbf{k}'}(\mathbf{I}),\Lambda^{-1}\mathbf{u}_{\mathbf{k}-\mathbf{k}'}(\mathbf{I}))$$

The second change of variables removes terms of the third order in ε depending on φ from the slow variable equations. In asymptotically small neighborhoods of the resonance surfaces

$$< \omega(\mathbf{I}), \mathbf{k} > = 0 \ (\mathbf{k} \in Z^n, |\mathbf{k}| \leq 2N)$$

the introduced changes of variables become meaningless. The properties of the solutions of system (8) at resonance must be investigated by the methods described in [11, Chapter III].

Far from the resonance surfaces the behavior of the slow variables with accuracy $O(\varepsilon)$ in the time interval ε^{-3} is described by the following evolution equations (we use the original notation for the averaged variables)

$$\mathbf{I}^{\bullet} = -\nabla_{\omega} \Phi_{eff}(\omega(\mathbf{I}), \mathbf{I}) \tag{9}$$

where

$$\Phi_{eff}(\omega, \mathbf{I}) = \frac{\varepsilon^3}{2} < \omega, D_{eff}\, \omega >$$

$$D_{eff} = << U_{\varphi} \Lambda^{-1} D \Lambda^{-1} U_{\varphi}^T >> = \sum_{\mathbf{k} \in Z^n, |\mathbf{k}| \leq N} (\mathbf{u}_{\mathbf{k}}, \Lambda^{-1} D \Lambda^{-1} \mathbf{u}_{-\mathbf{k}}) \mathbf{k}^T \mathbf{k}$$

$$<< \cdot >> = \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} (\cdot)\, d\varphi_1 \ldots d\varphi_n$$

The quadratic form $\Phi_{eff}(\omega, \mathbf{I})$ in (9) is an analogue of the function $\Phi(\mathbf{v}, \varepsilon)$ in (1) and describes the dissipation of energy in quasi-steady motion

$$<< \Phi(\mathbf{v}_*(\mathbf{I}, \varphi, \varepsilon), \varepsilon) >> = \Phi_{eff}(\omega(\mathbf{I}), \mathbf{I}) + O(\varepsilon^4)$$

5. THE $S_H + S_D$ SYSTEM AS A MODEL OF A DEFORMABLE SOLID PERFORMING TRANSLATIONAL-ROTATIONAL MOTION

In many investigations, for example, when studying the dynamics of large space structures or the tidal evolution of planetary rotation [3,12,13], the question arises of the translational-rotational motion of a deformable body in a potential field.

The motion of a deformable body with respect to its centre of mass consists of the rotation of the body as a whole and the elastic displacements $\mathbf{s}$ of its individual elements. The dissipation of mechanical energy during relative displacements leads to the damping of high-frequency normal oscillations and influences the motion of the body as a whole.

As a rule, the decay time of the natural oscillations is considerably less than the characteristic time of the motion of the body as a whole. Hence quasi-steady motion is fundamental for a deformable body.

We say that the system $S_H + S_D$ is an N th order model if the subsystem S_H describes the motion of the body as a whole taking no account of deformation, while the subsystem S_D describes the deformation of the body on the basis of a finite-dimensional approximation of the deformation field, using forms of free oscillation corresponding to the N lowest frequencies of the body.

As $N \to \infty$ the right-hand sides of equations for quasi-steady motion for models of corresponding order form a rapidly converging functional series. This enables us to consider low-order models for a qualitative analysis of the influence of deformations on the motion of specific objects.

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