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SURFACE WAVES IN A FLUID ANNULUS WITH A VIBRATING INNER CYLINDER

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ABSTRACT

The phenomenon of excitation of fluid free-surface waves between two cylindrical shells when the inner wall vibrates is investigated. To obtain a lucid picture of energy transmission from the wavemaker motion (inner shell vibrations) to the fluid free-surface motion the method of superposition has been used. In order to do this the fluid potential ϕ was presented as the sum of three harmonic functions and all eigenmodes for linear mathematical task were studied in details for any type of an excitation.

INTRODUCTION

The phenomenon of deterioration of fluid free-surface waves between two cylindrical shells when the inner wall vibrates radially, is rather known [1]. The waves may be excited by harmonic axisymmetric deformations of the inner shell and depending on the vibration frequency both axisymmetric and non-symmetric wave patterns may arise. Experimental observations have revealed that waves are excited in two different resonance regimes [2]. The first type of waves corresponds to forced resonance, in which axisymmetric patterns are realized with eigenfrequencies equal to the frequency of excitation. The second kind of waves are parametric resonance wavesand in this case the waves are "transverse", with their crests and troughs aligned perpendicular to the vibrating wall. These so-called cross-waves have frequencies equal to half of that of the wavemaker [3].

Garrett [4] has shown how energy is transferred from the wavemaker to the cross-wave in a mathematical model including a mean motion of the free surface. He mentioned, however, that this mean motion of the free surface is not sufficient to supply the energy to the cross-waves. Therefore, the cross-waves must derive their energy in some way directly from the wavemaker. To show direct transmission of energy from the wavemaker the method of superposition has been used and the fluid potential ϕ was presented as the sum of three harmonic functions.

1. TEORETICAL ANALYSIS

One considers theoretically the nonlinear problem of fluid free-surface waves which are excited by inner shell vibrations in a volume between two cylinders of finite length. It is useful to relate the fluid motion to the cylindrical coordinate system (r,θ,x). The fluid has an average depth d; the average position of the free surface is taken as $x=0$, so that the solid tank bottom is at $x=-d$.

The fluid is confined between a solid outer cylinder at $r=R_2$ and a deformable inner cylinder at average radius $R_1 = r_1 + a_0(d)^{-1}\int_{-d}^{0}\cos(\eta x)dx = r_1 + 2a_0/\pi$. This inner cylinder acts as the wavemaker and vibrates harmonically in such a way that the position of the wall of the inner cylinder is $r = R_1 + \chi_1(x,t) = R_1 - (a_0 + a_1\cos\omega t)\cos\eta x - 2a_0/\pi$, where $\eta = \pi/(2d)$. Assuming that the fluid is inviscid and incompressible, and that the induced motion is irrotational, the velocity field can be written as $\mathbf{v} = \nabla\phi$, with $\phi(r,\theta,x,t)$ the velocity potential. The governing equation is

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$$\nabla^2\phi = 0 \qquad \text{on} \qquad (R_1 + \chi_1 \le r \le R_2, 0 \le \theta \le 2\pi, -d \le x \le \zeta) \qquad (1)$$

where $\zeta(r,\theta,t)$ is free surface displacement.

The dynamic and kinematic free-surface boundary conditions are:

$$\phi_t + 1/2(\nabla\phi)^2 + g\zeta = \frac{T}{\rho}[\nabla^2\zeta - \frac{1}{2}\nabla\zeta\cdot((\nabla\zeta)^2\ \nabla\zeta)] + F(t) \qquad \text{at} \qquad x = \zeta(r,\theta,t) \qquad (2)$$

$$\phi_x = \nabla\phi\cdot\nabla\zeta + \zeta_t \qquad \text{at} \qquad x = \zeta(r,\theta,t) \qquad (3)$$

with g the gravitational acceleration, T the air-fluid surface tension and ρ the fluid density, $F(t)$ is an arbitrary function of time [5]. Here and later the subscripts x, r, θ, t signify partial differentiation.

The normal velocity vanishes at the solid flow boundaries:

$$\phi_r = 0 \qquad \text{at} \qquad r = R_2$$
$$\phi_x = 0 \qquad \text{at} \qquad x = -d$$

while the kinematic condition at the vibrating inner cylinder is:

$$\phi_r = \chi_t + \nabla\phi\cdot\nabla\chi_1 \qquad \text{at} \qquad r = R_1 + \chi_1(x,t). \qquad (4)$$

Effects of the meniscus and capillarity at the contact line of the fluid's free-surface and the annular container walls were not incorporated in the formulation of the problem. We assume, that

$$\zeta_r = 0 \qquad \text{at} \qquad r = R_1 \qquad \text{and} \qquad r = R_2$$

From the experimental observations we may conclude that the pattern formation has a resonance character, every pattern having its "own" frequency.

Assuming that patterns can be described in terms of normal modes with characteristic eigenfrequencies, we expand the potential ϕ and the free-surface displacement ζ in a complete set of eigenfunctions, which are determined by linear theory. The amplitudes of these eigenfunctions are governed by the nonlinear problem (2) - (3).

The solution of the linear general non-axisymmetric boundary problem

$$\nabla^2\phi = 0 \qquad \text{on} \qquad (R_1 \le r \le R_2, 0 \le \theta \le 2\pi, -d \le x \le 0), \qquad (5)$$

$$\begin{array}{llll}
\phi_r = \chi_t & \text{at} & x = 0 & (a) \\
\phi_x = 0 & \text{at} & x = -d & (b) \\
\phi_r = 0 & \text{at} & r = R_2 & (c) \\
\phi_r = \chi_t & \text{at} & r = R_1 & (d) \\
\phi_\theta\big|_{\theta=0} = \phi_\theta\big|_{\theta=2\pi} & & & (e)
\end{array} \qquad (6)$$

under arbitrary excitation of the inner cylindrical shell $w(\theta,x,t)$ can be found in several ways. One is Grinberg's method [6]. Here the potential ϕ is presented as Fourier series of the complete system of eigenfunctions in the radial and azimuthal coordinates with the coefficients as functions of the coordinate x. The inhomogeneous boundary condition at $r = R_1$ is transformed into the right-hand side of the equation (5) due to the ordinary procedure of the Fourier series representation for the derivatives on r. The solutions of the sequence of the inhomogeneous linear differential equations in x for the expansion coefficients with inhomogeneous boundary conditions in x can be easily found by any analytical techniques. This approach yields, however, rather cumbersome expressions, in which the input of the wavemaker motion $w(\theta,x,t)$ is not seen in a clear way. In order to obtain a more lucid picture of the transmission from the wavemaker motion to the free-surface motion it is more convenient to use another analytical method, namely, the method of superposition. The authors are of the opinion that the application of this method is without doubt preferable for the problem in question. It provides a clear physical picture of the mechanism of energy transfer from the wavemaker to the mean level variation and every eigenmode of free-surface oscillations. The idea of the superposition method was first proposed by Lamé in his classical lectures on the theory of elasticity [7].

According to this superposition method, the potential ϕ can be written as the sum of three harmonic functions:

$$\phi = \phi_0 + \phi_1 + \phi_2. \tag{7}$$

The potential ϕ_0 is governed by the following axisymmetric boundary problem:

$$\nabla^2 \phi_0 = 0 \qquad \text{on} \qquad (R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi,\ -d \le x \le 0) \tag{8}$$

$$\begin{array}{lll} (\phi_0)_x = (\zeta_0)_t & \text{at} \quad x = 0 & (a) \\ (\phi_0)_x = 0 & \text{at} \quad x = -d & (b) \\ (\phi_0)_r = 0 & \text{at} \quad r = R_2 & (c) \\ (\phi_0)_r = (w_0)_t & \text{at} \quad r = R_1 & (d) \\ (\phi_1)_\theta |_{\theta=0} = (\phi_1)_\theta |_{\theta=2\pi} & & (e) \end{array} \tag{9}$$

where

$$\begin{aligned} \zeta_0(t) &= \frac{1}{\pi(R_2^2 - R_1^2)} \int_0^{2\pi} \int_R^{R_2} \zeta(r,\theta,t) r dr d\theta \\ w_0(t) &= \frac{1}{2\pi R_1 d} \int_0^{2\pi} \int_{-d}^{0} w(\theta,x,t) R_1 dx d\theta \end{aligned} \tag{10}$$

represents the mean level elevation of the fluid free surface and mean displacement of the cylindrical wavemaker, respectively. These mean values are connected by the relationship

$$(\zeta_0)_t \pi(R_2^2 - R_1^2) - 2\pi d R_1 (w_0)_t = 0 \tag{11}$$

expressing mass conservation for the incompressible fluid. Thus, for the particular case of the wavemaker excitation it is easy to derive from this relationship the mean level oscillation:

$$\zeta_{00}(t) = \frac{4R_1 d}{\pi(R_2^2 - R_1^2)}(a_1 \cos \omega t). \tag{12}$$

The potential ϕ_1 is governed by the following linear problem:

$$\nabla^2 \phi_1 = 0 \qquad \text{on} \qquad (R_1 \le r \le R_2, 0 \le \theta \le 2\pi, -d \le x \le 0) \tag{13}$$

$$\begin{array}{lll} (\phi_1)_x = (\zeta - \zeta_0)_t & \text{at} \quad x = 0 & (a) \\ (\phi_1)_x = 0 & \text{at} \quad x = -d & (b) \\ (\phi_1)_r = 0 & \text{at} \quad r = R_2 & (c) \\ (\phi_1)_r = 0 & \text{at} \quad r = R_1 & (d) \\ (\phi_1)_\theta \big|_{\theta=0} = (\phi_1)_\theta \big|_{\theta=2\pi} & & (e) \end{array} \tag{14}$$

where the conditions in the radial direction are homogeneous and in the azimuthal direction periodic. So ϕ_1 will be expressed as a sum of complete systems of eigenfunctions in the radial and azimuthal coordinates.

While the potential ϕ_2 is governed by

$$\nabla^2\phi_2 = 0 \qquad \text{on} \qquad (R_1 \le r \le R_2, 0 \le \theta \le 2\pi, -d \le x \le 0), \tag{15}$$

$$\begin{array}{lll} (\phi_2)_x = 0 & \text{at} \quad x = 0 & (a) \\ (\phi_2)_x = 0 & \text{at} \quad x = -d & (b) \\ (\phi_2)_r = 0 & \text{at} \quad r = R_2 & (c) \\ (\phi_2)_r = (w - w_0)_t & \text{at} \quad r = R_1 & (d) \\ (\phi_2)_\theta \big|_{\theta=0} = (\phi_2)_\theta \big|_{\theta=2\pi} & & (e) \end{array} \tag{16}$$

it can be represented as a sum of eigenfunctions in the vertical (homogeneous boundary conditions (16*a*) and (16*b*)) and in the circumferential (condition of periodicity (16*e*)) directions. (The potential ϕ_2 does not cause any changes in the velocity of the displacement ζ. at the surface. However, it provides the pressure component which "supports" the free-surface motion, as can be seen from (2). This component has an excitation frequency equal to the frequency of $w(\theta, x, t)$ in the linear approximation of the problem.)

It is worth noting that the boundary problems (8)-(9), (13)-(14) and (15)-(16) are of the Neumann type when the normal derivative of the harmonic function is prescribed. For the solutions without singularities in the corner points Green's second theorem requires that these prescribed values should satisfy the condition of zero flux across the boundary. Obviously, this property is satisfied for all three boundary problems.

The solution of the boundary problem for ϕ_0 can be easily found as

$$\phi_0(r,\theta,t) = -\dot{w}_0(t)\frac{R_1}{R_2^2 - R_1^2}\left(\frac{r^2}{2} - R_2^2 \ln r\right) + \dot{\zeta}_0 \frac{(d+x)^2}{2d} \tag{17}$$

(here the dot means the time derivative), which identically satisfies the Laplacian equation (8) due to the relation (11).

The solution of the linear problem (13)-(14) for ϕ_1 can be written in the form

$$\phi_1 = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \phi_{ij}^{c,s}(t) \frac{\cosh k_{ij}(x+d)}{N_{ij} \cosh k_{ij} d} \psi_{ij}^{c,s}(r,\theta), \tag{18}$$

on the complete systems of azimuthal ($\cos i\theta$, $\sin i\theta$), and radial eigenfunctions

$$\chi_{ij}(k_{ij} r) = J_i(k_{ij} r) - \frac{J_{i'}(k_{ij} R_1)}{Y_{i'}(k_{ij} R_1)} Y_i(k_{ij} r)$$

with some arbitrary amplitudes $\phi_{ij}^{c,s}(t)$.

In the solution (18) the notations

$$\psi_{ij}^{c,s}(r,\theta) = \chi_{ij}(k_{ij} r)(\cos i\theta, \sin i\theta) \tag{19}$$

are used, where J_i and Y_i are the i-th order Bessel functions of the first and the second kind, respectively, and N_{ij} is a normalization constant implied from the relation $N_{ij}^2 = \int_0^{2\pi}\int_{R_1}^{R_2} (\psi_{ij}^{c,s})^2 r dr d\theta$, where the index c (or s) indicates that the eigenfunction $\cos i\theta$ (or $\sin i\theta$) is chosen as the circumferential component; k_{ij} represents the roots of the equation

$$J'_i(k_{ij}R_2) - \frac{J'_i(k_{ij}R_1)}{Y'_i(k_{ij}R_1)} Y'_i(k_{ij}R_2) = 0\ .$$

The system of functions $\psi_{ij}(r,\theta)$, with $i = 0,1,2,...$ and $j = 1,2,3,...$, is a complete orthogonal system, so any function of the variables r and θ can be represented using the usual procedure of Fourier series expansion.

Thus, the free surface displacement $\zeta(r,\theta,t) - \zeta_0(t)$ can be written as

$$\zeta(r,\theta,t) - \zeta_0(t) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \zeta_{ij}^{c,s}(t) \frac{\psi_{ij}^{c,s}(r,\theta)}{N_{ij}}. \tag{20}$$

The boundary condition (14*a*) provides the relation between the amplitudes of the series (18) and (20) in the form:

$$\phi_{ij}^{c,s}(t) = \dot{\zeta}_{ij}^{c,s}(t)(k_{ij} \tanh k_{ij} d)^{-1} \tag{21}$$

The velocity potential $\phi_2(r,\theta,x,t)$ can be formulated in terms of an ordinary Fourier series in $\cos\alpha_l x$ with $\alpha_l = l\pi / d$ and in ($\cos i\theta$, $\sin i\theta$), so that the general solution reads

$$\phi_2 = \sum_{i=0}^{\infty}\sum_{l=1}^{\infty} \Phi_{il}^{c,s}(t) \cos\alpha_l x \hat{\chi}_{il}(\alpha_l r)(\cos i\theta, \sin i\theta) \tag{22}$$

with

$$\hat{\chi}_{il}(\alpha_l r) = I_i(\alpha_l r) - \frac{I'_i(\alpha_j R_2)}{K'_i(\alpha_j R_2)} K_i(\alpha_l r)$$

were I_i and K_i the i-th order modified Bessel functions of the first and second kind, respectively.

Using the boundary condition (16*d*) we can explicitly define the amplitudes $\Phi_{il}^{c,s}(t)$ as

$$\Phi_{il}^{c,s}(t) = \dot{w}_{il}^{c,s}(t) = \frac{2 - \delta_{i0}}{d\pi\alpha_l \hat{\chi}_{il'}(\alpha_l R_1)} \int_0^{2\pi}\int_{-d}^{0} [\dot{w}(\theta,x,t) - \dot{w}_0(t)] \cos\alpha_l x (\cos i\theta, \sin i\theta) R_1 \, dx \, d\theta,$$

where δ_{i0} is the Dirac function and $\hat{\chi}_{0l'}(z) = d\,\hat{\chi}_{0l}(z)/dz$.

To define the unknown functions $\zeta_{ij}^{c,s}(t)$, representing the amplitudes of directly excited free surface waves, we have to apply the linearized dynamic free-surface boundary condition (2)

$$\phi_t + g\zeta - \frac{T}{\rho}\nabla^2\zeta = F(t) \text{ at } x = 0, \tag{23}$$

where ϕ represents the total velocity potential according to (7).

Substitution of (7) into (23) leads to the functional equation on r in the interval (R_1, R_2). Representing the radial functions $r^2/2 - R_2^2 \ln r$ and $\hat{\chi}_{il}(\alpha_l r)$ in the form of the expansions

$$\frac{r^2}{2} - R_2^2 \ln r = a_{00} + \sum_{j=1}^{\infty} a_{0j} \frac{\chi_{0j}(k_{0j} r)}{N_{0j}}$$

$$\hat{\chi}_{il}(\alpha_l r) = b_{il0} + \sum_{j=1}^{\infty} b_{ilj} \frac{\chi_{ij}(k_{ij} r)}{N_{ij}},$$

where the coefficients a_{00}, a_{0j}, b_{i0l} and b_{ilj} can be found by straightforward integration, we can write down the infinite sequence of ordinary differential equations for the functions $\zeta_{ij}^{c,s}(t)$:

$$\begin{cases} \ddot{\zeta}_{0j}(t) + \omega_{0j}^2 \zeta_{0j}(t) = \ddot{w}_0(t) \dfrac{a_{0j}\beta_{0j}R_1}{(R_2^2 - R_1^2)} - \sum_{l=1}^{\infty} \ddot{w}_{0l}^c(t) b_{0lj}\beta_{0j}, \\ \ddot{\zeta}_{ij}^{c,s}(t) + \omega_{ij}^2 \zeta_{ij}(t) = -\sum_{l=1}^{\infty} \ddot{w}_{il}^{c,s}(t) b_{ilj}\beta_{ij}, \end{cases} \tag{24}$$

where $\beta_{ij} = k_{ij} \tanh k_{ij} d$ and $\omega_{ij} = \left[(gk_{ij} + \dfrac{T}{\rho} k_{ij}^3) \tanh k_{ij} d \right]^{1/2}$ for $i = 0,1,2,...$ and $j = 1,2,3,...$

The linear equations (24) represent typical equations of the forced oscillations with eigenfrequencies ω_{ij}. Solving these linear differential equations with specified initial conditions under prescribed time dependence of the functions $w_0(t)$ and $w_{il}^{c,s}(t)$, we can easily obtain the amplitudes
$\zeta_{ij}(t)$ of the fluid free-surface waves in an explicit manner.

CONCLUSIONS

A simple mathematical model, which shows how the cross-wave can be generated directly by the wavemaker motion without having to take into account the presence of any axisymmetric waves at the free surface. This mathematical model of the excitation of the resonant cross-waves may be the easiest way to understand pattern formation on the fluid's free surface

The nonlinear problems for resonant eigenmodes could be solved in the following way. First, for finding the amplitudes of the potential ϕ_2 the nonlinear boundary condition (4) is applied with the expansion procedure in the series with $\cos\alpha_l x$ and $(\cos i\theta,\ \sin i\theta)$ functions. The second step is to determine the relations between the amplitudes of potential ϕ_1, the functions $\phi_{ij}(t)$ and the amplitudes $\zeta_{ij}(t)$ of the fluid free-surface waves according to the nonlinear boundary condition (3). And finally, the dynamic condition (2) should be taken in consideration for the closure step, namely, to obtain nonlinear differential equations for resonant amplitudes under the prescribed excitation.

REFERENCES

[1] Becker, J.M., Miles, J.W. Standing radial cross-waves *Journal of Fluid Mechanics,* Vol. 222., pp. 471-499, 1991.

[2] Krasnopolskaya, T.S., van Heijst, F.G.J. Wave pattern formation in a fluid annulas with a vibrating inner shell *Journal of Fluid Mechanics,* Vol. 328, pp. 229-252, 1996.

[3] Faraday, M. On a peculiar class of acoustical figures; and on certain forms assumed by groups of particles upon vibrating elastic surface, *Philosophical Transactions of Royal Society of London A,* Vol. 121, pp. 299-340, 1831.

[4] Garrett, C. J. R. Cross waves *Journal of Fluid Mechanics,* Vol. 41, pp. 837-849, 1970.

[5] Lamb, H. *Hydrodynamics*, Cambridge University Press, Cambridge, 1932.

[6] Lebedev, N. N., Skal'skaya, I. P., Uflyand, Ya. S. *Problems in Mathematical Physics*. Pergamon, Oxford, 1966.

[7] Lamé, G. *Leçons sur la Théorie Mathématique de l'Élasticité des CorpsnSolids* Bachelier, Paris, 1852.