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THE BIFURCATION SET FOR A TWO-AXES VEHICLE MODEL WITH THE NON-LINEAR DEPENDENCE OF SLIPPING FORCES

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ABSTRACT

The approach to construct the bifurcation set of steady states for a two-axes vehicle model considering non-linear non-monotone dependences of slipping forces, is presented. Phase portraits illustrating cases of stability loss are given.

1. THEORETICAL GROUNDING.

The mathematical model of a vehicle can be presented as the dynamic system of the form:

$$\dot{x} = f\left(x, \theta, v\right). \tag{1}$$

Its steady states result from the solution of the following non-linear equations

$$f_i(x, \theta, v) = 0, \quad x \in R^n, \quad (i = 1, ..., n). \tag{2}$$

The system has two control parameters: longitudinal motion velocity v and the turning angle θ of the front steering wheels.

In papers [1, 2] the steady states evolution resulted from the variations of control parameters is analyzed.

Bifurcation values (v^*, θ^*) correspond to multiple solutions x^* of the system (2).

Jacobian system is altered to zero at all points of the critical set x^*:

$$J = \left\| \partial f_i / \partial x_j \right\|_{x^*} = 0, \qquad x^* \in M_{kp}.$$

The system (2) with the above-mentioned equation gives rise to the critical set on the basis of the steady states manifold.

At critical points of the set the steady stationary state is eliminated (these points correspond to either fold – two-fold system solutions (2), or cusp – three-fold system solutions (2)).

Any qualitative variations of stationary states for the system of control parameters result from the birth (elimination) of two singularities.

Therefore the determination of stability boundaries considering the control parameters is of interest.

Stability boundaries can be defined by constructing the bifurcation set which divides the parameters domain into a number of domains with different stationary states, determining the zones of stability or instability.

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However, the procedure of constructing the bifurcation set in the case of dependences of sideways slipping forces with evident maximum demands a further development. Such necessity is explained by the existence of several branches of the bifurcation set.

2. FORMULATION OF THE PROBLEM.
THE CONSTRUCTION OF THE BIFURCATION SET FOR A TWO-AXES VEHICLE MODEL.

Let's us analyze the system consisting of the body with a rear wheel axis firmly fixed and the front wheel module, its turning about the body is rigidly fixed (set by *θ*).

The system is subjected to the sideways reaction of the support plane – elastic wheels can move at some angle (slipping angle) to the surface of the wheel symmetry (due to elastic deformation at the point of the contact). Such situation gives rise to transverse forces resulted from the supporting plane thus interfering with sideways slipping of the wheel (slipping forces).

Let *m* is the vehicle mass; *J* is the central inertia moment of the system about the vertical axis; *a*, *b* are distances between the centre of mass of the vehicle to the middle part of the front and rear wheel axes correspondently.

Equations of the plane-parallel motion for the velocipede two-axes scheme vehicle (vertical longitudinal plane across the middle of wheel axes is the plane of the symmetry) with the constant longitudinal constituent of the mass centre velocity are

$$\begin{cases} m(\dot{u}+\omega v) = Y_1\cos\theta + Y_2; \\ J\dot{\omega} = aY_1\cos\theta - Y_2 b; \\ \delta_1 = \theta - arctg\dfrac{u+a\omega}{v},\ \delta_2 = arctg\dfrac{-u+b\omega}{v}. \end{cases} \tag{3}$$

where *u* is the transverse constituent of the vehicle's mass centre velocity; *ω* is the angle velocity of the vehicle about the vertical axis; δ_1, δ_2 are the slipping angles for front and rear axes correspondently; Y_1 , Y_2 are the sideways slipping forces as functions of slipping angles for front and rear axes correspondently.

Slipping forces are defined empirically and can be represented through different analytical dependences:

$$\begin{aligned} &-\frac{v}{g}\omega + \frac{\cos(\theta)\overline{Y_1}(\delta_1)b}{l} + \frac{\overline{Y_2}(\delta_2)a}{l} = 0, \\ &\cos(\theta)\overline{Y_1}(\delta_1) - \overline{Y_2}(\delta_2) = 0. \end{aligned} \tag{4}$$

In this case the determination of steady motion states (singularities) has the form

Where $\overline{Y_i}(\delta_i) = Y_i(\delta_i)/N_i$ – dimensionless sideways reactions of the support plane on the axis (N_i – vertical load on the axis).

In our paper we deal with dependences of the type

$$Y_i = \frac{\gamma_i\delta_i}{\sqrt{1+\dfrac{\left(|\delta_i|-\beta_i\right)^2}{(\beta_i)^2}}}, \tag{5}$$

which guarantee the nonmonotonicity of slipping forces (unlike monotone dependences at considerable slipping angle the function has descending sections).

Parameters γ_i and β_i are due to keeping geometrical characteristics of the monotone dependences, $Y_i = \dfrac{q_i\delta_i}{\sqrt{1+\dfrac{(q_i\delta_i)^2}{(\varphi_i)^2}}}$, enabling the constancy of the critical velocity for rectilinear motion, coordination of maximum values of dimensionless slipping forces (Fig.1):

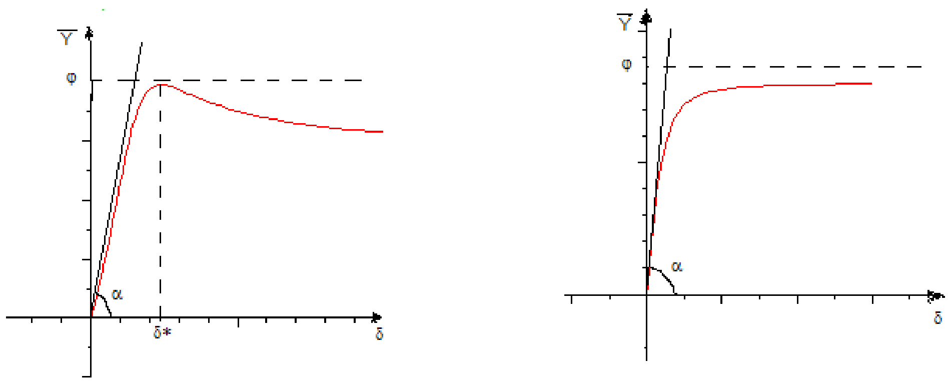

Fig.1. Non-monotone and monotone dependences of slipping angles

One analyzes the influence of the new “geometry” of slipping forces dependences on the bifurcation set.

Earlier (for monotone dependences) different types of bifurcation sets were obtained within “geometric” approach.

The original system determining the steady states has the form

$$\bar{Y}(\delta_2-\delta_1)=\frac{v^2}{gl}(\theta+\delta_2-\delta_1), \quad (6)$$

where the left part of the equation is a non-linear function and named “stationary curve”, the right part of the equation presents a straight line (“moving line”).

The intersection points of the “stationary curve” and “moving line” correspond to stationary states of the system (2).

Parameters v and θ being constantly changed, the equation (6) sets the reflection of the plane with v and θ to balanced surface.

The bifurcation set (critical set) corresponds to v, θ for which “the moving straight line” contacts with “the stationary curve”.

Points of the inflection of the original curve $\bar{Y}=\bar{Y}(\delta_2-\delta_1)$ correspond to the points of the bifurcation set cusp.

The triple solution for the balanced plane is corresponded to the cusp, double solution – to the fold.

In the case of the monotone dependences of slipping forces from slipping angles of the saturation curve, the “stationary curve” can have three points of inflection, the bifurcation set – three cuspidal points.

The symmetric “cusp” corresponds to the three-fold steady state at $v = v_{\kappa p}^{+}$, and $\theta = 0$ (the stability loss for rectilinear motion), where $V_{\kappa p}=\sqrt{\frac{glq_1q_2}{q_1-q_2}}$, $q_i=\frac{k_i}{N_i}$ are stationary dimensionless slipping coefficients [4].

In the case of descending original dependences $\bar{Y}_i(\delta_i)$ additional points of inflection of the “moving curve” $\bar{Y}=\bar{Y}(\delta_2-\delta_1)$ come into being, resulting in the complication of the bifurcation set.

Let’s analyze the method of constructing the bifurcation set for definite numeric values of γ, β:

$$Y_1=\frac{\delta\cdot 3.300062959\cdot\sqrt{2}}{\sqrt{1+\frac{(|\delta|-0.12)^2}{0.12^2}}},\qquad Y_2=\frac{\delta\cdot 2.526513230\cdot\sqrt{2}}{\sqrt{1+\frac{(|\delta|-0.15)^2}{0.15^2}}},\qquad \delta\in[-1;1].$$

The dependence $\bar{Y}=\bar{Y}(\delta_2-\delta_1)$ is determined by $\bar{Y}_1(\delta_1)=\bar{Y}(\delta_2)=\bar{Y}$.

Critical values of v, θ correspond to the next equations (7)

$$\frac{v^2}{gl} = \frac{dY}{d(\delta_2 - \delta_1)};$$
$$\frac{Y}{\theta + \delta_2 - \delta_1} = \frac{dY}{d(\delta_2 - \delta_1)}, \tag{7}$$

Then

$$\theta = Y \cdot Y' - (\delta_2 - \delta_1). \tag{8}$$

Therefore the system (7) gives rise to the bifurcation set having the parametric form

$$\theta = \theta(\delta_2 - \delta_1), \quad v = v(\delta_2 - \delta_1). \tag{9}$$

Sometimes Y as a parameter is more preferable then $(\delta_2 - \delta_1)$. The original dependences are $Y_1 = f_1(\delta_1)$, $Y_2 = f_2(\delta_2)$. Solving them with respect to δ_i, we can find $\delta_1 = F_1(Y_1)$, $\delta_2 = F_2(Y_2)$. Therefore, $G(Y) = F_2(Y_2) - F_1(Y_1)$. In this case the final version of the equation (6) is

$$\frac{gl}{v^2} \cdot Y - \theta = G(Y). \tag{10}$$

Following the contact (Fig. 3, b) of the “stationary curve” and “moving straight line” as

$$\frac{gl}{v^2} = \frac{dG}{d(Y)};$$
$$\frac{\theta + G(Y)}{Y} = \frac{dG}{d(Y)}, \tag{11}$$

we can obtain parametric equations of the bifurcation set in the form $\theta = \theta(Y)$, $v = v(Y)$

$$\theta = Y \cdot G'(Y) - G(Y);$$
$$v = \sqrt{\frac{gl}{G'(Y)}}. \tag{12}$$

Then the procedure of forming function $G(Y) = \delta_2 - \delta_1$ in the case of non-monotone dependences $Y_i(\delta_i)$ is analyzed.

For numerical values of β, γ we define functions $F_i(Y)$, admitting the correlation (5) for δ_i, we have two single-valued branches, connected at the points of turning (Fig. 2):

$$f_{11} = \frac{0.12\left(-|Y| + \sqrt{0.6272879344 - Y^2}\right)Y}{-Y^2 + 0.3136439672}, \quad f_{21} = \frac{0.15\left(-|Y| + \sqrt{0.5744942192 - Y^2}\right)Y}{-Y^2 + 0.2872471096},$$

$$f_{12} = \frac{0.12\left(Y + \sqrt{0.6272879344 - Y^2}\right)Y}{Y^2 - 0.3136439672}, \quad f_{22} = \frac{0.15\left(Y + \sqrt{0.5744942192 - Y^2}\right)Y}{Y^2 - 0.2872471096}.$$

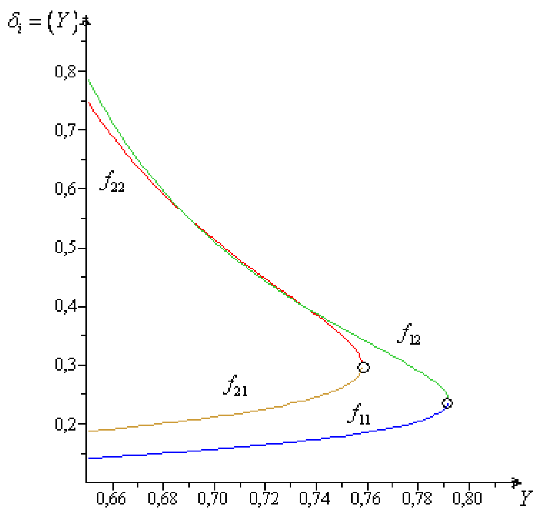

Fig. 2. Dependences of sideways slipping forces at γ_1=3.30006295, β_1=0.12 and γ_2=2.52651323, β_2=0.15.

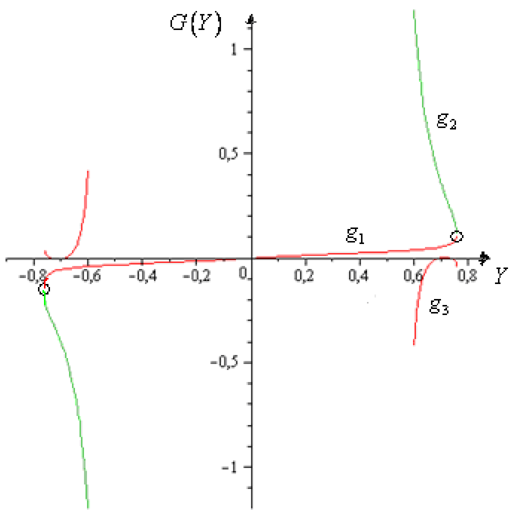

Fig. 3. Chart of stationary curve for selected γ, β.

Therefore, the function $G(Y) = \delta_2 - \delta_1$ is determined as the difference of corresponding single-valued branches as f_{ij}, and has three branches of single-valuedness $\{g_1, g_2, g_3\}$; g_1 and g_2 are connected at the point of turning, thus forming the "main" branch (Fig.3).

The section of the main branch up to the point of turning comes from $G(Y) = g_1 = f_{21} - f_{11}$, the second part of this branch has the form $G(Y) = g_2 = f_{22} - f_{11}$.

The additional branch of the "moving curve" is due to the descending sections of slipping forces dependences $G(Y) = g_3 = f_{22} - f_{12}$.

Every section of the function $G(Y)$ in accordance with (12) has a dual curve, presenting the part of the bifurcation set (Fig.4).

The bifurcation set divides the plane of control parameters θ, ν into domains with different number of stationary states. It is also possible to determine the number of steady and unsteady states for each domain. The critical set of parameters being intersected, the number of stationary states is changed into two states. The number of stationary states in different domains with the control parameters plane is illustrated in Fig.4.

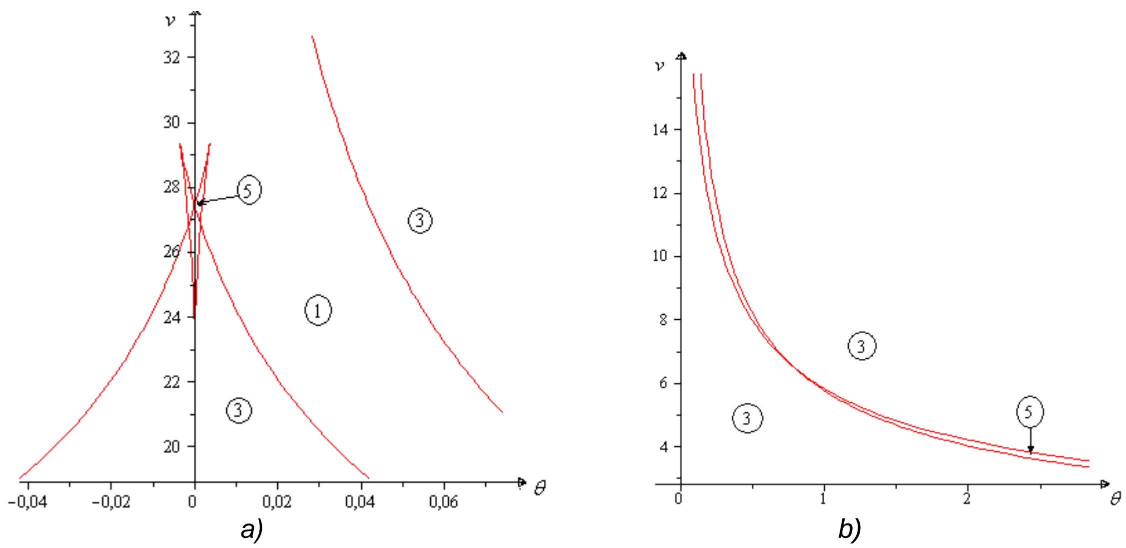

Fig. 4. The bifurcation set (non-linear dependence from a slipping angle) without "heel" moment: a) general set view, b) fragments of the set.