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TWO-MECHANISM MODELS AND MODELLING OF CREEP

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ABSTRACT

Two-mechanism models (or, generally, multi-mechanism models) are a useful tool for modelling of complex material behaviour, in particular for modelling of interaction of creep and plasticity. As we will demonstrate, pure creep can also be modelled by two-mechanism models.

INTRODUCTION

1) Two-mechanism (or, generally, multi-mechanism) models have been studied and applied for the last twenty years. Their characteristic trait is the additive decomposition of the inelastic (i.e., plastic or visco-plastic, e.g.) strain into two (or multi) parts (sometimes called ``mechanisms") in the case of small deformations. In comparison with rheological models (cf. [1], e.g.), there is an interaction between these mechanisms (see Figure 1). This interaction allows to describe important observable effects, but, it requires additional efforts in modelling and simulation. Each inelastic strain part may exhibit plastic, creep or general inelastic behaviour. The (thermo-)elastic strain is not regarded as an own mechanism. Each mechanism has its own internal variables with corresponding evolution equations. Moreover, each mechanism may have an own yield criterion, or, there may be common yield criteria for several mechanisms. Thus, in the case of two mechanisms, there are possible models of the type 2M1C and 2M2C. That means two mechanisms with one or two yield criteria. A mechanism without yield criterion like creep can be formally treated as a mechanism with its own criterion with zero yield stress.

If the inelastic strain is seen as one mechanism (as it was historically first), one refers to a “unified model” (or “Chaboche” model) (cf. the survey [2] and the references cited therein). (That means plastic and viscous components are considered together in the same variable.) As explained in [3] and [4], there are experimentally observable effects (inverse strain-rate sensibility, e.g.) which can be qualitatively correctly described by the two-mechanism approach.

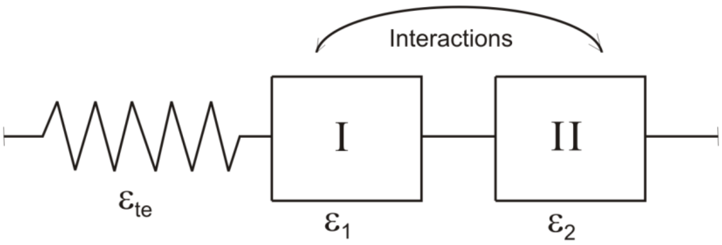

Fig. 1 Scheme of a two-mechanism model. The two inelastic mechanisms 1 and 2 have their own evolution equations. But, they are not independent from each other. The thermoelastic strain ε_{te} is usually not regarded as a mechanism

2) For modelling and applications of multi-mechanism models we refer to [3], [4], [5], [6], [2], [7], [8], [9], [10] and the references therein.

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3) Two-mechanism models have been applied in modelling of cyclic plasticity (cf. [7], [8],[10], e.g.) and of steel behaviour (cf. [11] and the references therein). Moreover, there is a large variety of papers dealing with complex material behavior of metals, soils, composites, biological tissues etc. in which the inelastic strain is decomposed into several parts. But, as a rule, multi-mechanism models are not directly addressed. In [9], some references can be found.

4) Creep is a complex phenomenon of material behaviour. Thus, there are several approaches of modelling (cf. [12]). To our knowledge, creep (alone) is not modelled in the framework of 2M models. In this note, we propose first steps for doing so.

1. SOME CLASSES OF TWO-MECHANISM MODELS

In short we provide important basic relations for 2M models. Due to the limitation of this extended abstract, we only deal with 2M2C models. Besides, these models can well describe possible interactions of plasticity and creep as well as creep alone.

1.1 General assertions

In the framework of small deformations, the balance equation of momentum and energy as well as the Clausius-Duhem inequality are given by

$$\rho\ddot{\mathbf{u}} - div\,\boldsymbol{\sigma} = \boldsymbol{f}, \quad \rho\dot{e} + div\,\boldsymbol{q} = \boldsymbol{\sigma}:\dot{\varepsilon} + r \tag{1}$$
$$-\rho\dot{\psi} - \rho\eta\dot{\theta} + \boldsymbol{\sigma}:\dot{\varepsilon} - \frac{1}{\theta}\cdot\nabla\theta \geq 0 \tag{2}$$

The relations (1) and (2) have to be fulfilled in the space-time domain $\Omega\times]0, T[$. The notation is standard: ρ - density in the reference configuration, that means for t = 0, $\mathbf{u}$ - displacement vector, $\boldsymbol{\varepsilon}$ - linearized Green strain tensor, θ - absolute temperature, $\boldsymbol{\sigma}$ - Cauchy stress tensor, $\mathbf{f}$ - volume density of external forces, e - mass density of the internal energy, $\mathbf{q}$ - heat-flux density vector, r - volume density of heat supply, ψ - mass density of free (or Helmholtz) energy, η - mass density of entropy. The time derivative is denoted by a dot. $\boldsymbol{\alpha}:\boldsymbol{\beta}$ is the scalar product of the tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $\mathbf{q}\cdot\mathbf{p}$ is the scalar product of the vectors $\mathbf{p}$ and $\mathbf{q}$. We note the well-known relations

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \qquad \psi = e - \theta\eta \tag{3}$$

In the general case of inelastic material behaviour, the full strain $\boldsymbol{\varepsilon}$ is split up via

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{te} + \boldsymbol{\varepsilon}_{in} \tag{4}$$

($\boldsymbol{\varepsilon}_{te}$ - thermoelastic strain, $\boldsymbol{\varepsilon}_{in}$ - inelastic strain). Usually, the inelastic strain is assumed to be traceless, i.e.

$$tr(\boldsymbol{\varepsilon}_{in}) \tag{5}$$

The accumulated inelastic strain is defined by

$$s_{in}(x,t) := \int_0^t (\frac{2}{3}\dot{\boldsymbol{\varepsilon}}_{in}(x,\tau):\dot{\boldsymbol{\varepsilon}}_{in}(x,\tau))^{\frac{1}{2}} d\tau \tag{6}$$

We propose for the free energy ψ the split

$$\psi = \psi_{te} + \psi_{in} \tag{7}$$

The thermoelastic part ψ_{te} is standard (cf. [9] for details) and leads to the usual material law connecting stress and thermoelastic strain:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}_{te}^* + Ktr(\boldsymbol{\varepsilon}_{te})\mathbf{I} - 3K\alpha(\theta - \theta_0)\mathrm{I} \tag{8}$$

$\mu > 0$ - shear modulus, $K > 0$ - compression modulus, α - linear heat-dilatation coefficient, θ_0 – initial temperature, i.e. t = 0, $\mathbf{I}$ – unity tensor, $\boldsymbol{\varepsilon}_{te}^*$ - deviator of $\boldsymbol{\varepsilon}_{te}$, defined (in 3d case) by

$$\boldsymbol{\varepsilon}_{te}^{*} = \boldsymbol{\varepsilon}_{te} - \frac{1}{3} tr(\boldsymbol{\varepsilon}_{te})\mathbf{I} \tag{9}$$

We assume that the inelastic part ψ_{in} of ψ has the general form

$$\psi_{in} = \psi_{in}(\xi, \theta) \tag{10}$$

$\xi = (\xi_1, ..., \xi_m)$ (ξ - scalars or tensors) represent the internal variables. Further on, these variables will be chosen in accordance with concrete models under consideration. In the case of damage, the thermoelastic part ψ_{te} of the free energy may depend on internal variables too (cf. [12], e.g.). Internal variables have to fulfil evolution equations which are usually ordinary differential equations (ODE) with respect to the time t. As a rule, one poses zero initial conditions, i.e.

$$\xi_j(0) = 0 \quad \text{for } j = 1, \dots, m \tag{11}$$

Using standard arguments of thermodynamics (cf. [12], [13], e.g.) and assuming Fourier's heat-conduction law, from (2) one obtains the remaining dissipation inequality:

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho \textstyle\sum_{j=1}^{m} \frac{\partial \psi_{in}}{\partial \xi_j} : \dot{\xi}_j \geq 0 \tag{12}$$

If (12) is fulfilled for arbitrarily chosen sets of variables, then the model under consideration is thermodynamically consistent.

Until now, the relations developed above are addressed to one-mechanism models (“Chaboche” models) as well as to two-mechanism models.

In the theory of 2M models the following decomposition is crucial:

$$\boldsymbol{\varepsilon}_{in} = A_1 \boldsymbol{\varepsilon}_2 + A_2 \boldsymbol{\varepsilon}_2 \tag{13}$$

A_1, A_2 are positive real numbers. As usual, the inelastic strains are trace-less:

$$tr(\boldsymbol{\varepsilon}_1) = tr(\boldsymbol{\varepsilon}_2) = 0 \tag{14}$$

Remark 1. (i) The parameters A_1 and A_2 open opportunities for further extensions and special applications. We refer to [5]. In many applications, A_1 and A_2 are equal to 1, but, they can depend on further quantities. For instance, they can constitute phase fraction in complex materials (steel, shape memory alloys, e.g.). In this sense, here is a bridge from the macro to the meso (or micro) level of modelling.
(ii) In case of n mechanisms, instead of (13), one has the split

$$\boldsymbol{\varepsilon}_{in} = \textstyle\sum_{j=1}^{m} A_j \boldsymbol{\varepsilon}_j \tag{15}$$

with $A_j > 0$. In this note, we preferably deal with 2M models.

For both $\boldsymbol{\varepsilon}_j$ we introduce *separate* accumulations

$$s_j(x,t) := \int_0^t \left(\frac{2}{3} \dot{\boldsymbol{\varepsilon}}_j(x,\tau) : \dot{\boldsymbol{\varepsilon}}_j(x,\tau)\right)^{\frac{1}{2}} d\tau, \;\; j = 1,2 \tag{16}$$

Note, that s_{in} (as defined in (6)) is *not* the sum of s_1 and s_2.

We introduce the local stresses $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ via

$$\boldsymbol{\sigma}_j = A_j \boldsymbol{\sigma} \tag{17}$$

To develop further the theory, 2M1C and 2M2C models are separately considered. As mentioned above, here, we only deal with 2M2C models.

1.2 Two-mechanism models with two yield criteria

To focus, here, we do not consider isotropic hardening in the case of (visco-)plastic mechanisms. Thus, the forthcoming explanations will become shorter. However, the main idea of the two-mechanism approach (mutual coupling of mechanisms) can be made clear. We refer to [9] and [10] for detailed descriptions.

The ansatz for the inelastic part of the free energy in (10) will be specialised in the following way: Assuming the internal variables to be given $\xi = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$, we suppose

$$\psi_{in} = \psi_{in}(\theta, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) := \frac{1}{3\rho}\left(c_{11}(\theta)\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_1 + 2c_{12}(\theta)\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 + c_{22}(\theta)\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2\right) \tag{18}$$

The tensorial symmetric internal variables of strain type $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are related to kinematic hardening and associated with the mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$, respectively.

Remark 2. For “frozen” temperature, the inelastic free energy ψ_{in} in (18) is a convex function with respect to $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, if there hold (for all admissible θ) the conditions

$$c_{11}(\theta) \geq 0, \qquad c_{12}^2(\theta) \leq c_{11}(\theta)c_{22}(\theta) \tag{19}$$

Clearly, the quadratic form in (18) is also positive semi-definite.

The definition of the backstresses $\mathbf{X}_1$ and $\mathbf{X}_2$ associated with the mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$, respectively, and (18) give

$$X_1 := \rho\frac{\partial \psi_{in}}{\partial \alpha_1} = \frac{2}{3}c_{11}\alpha_1 + \frac{2}{3}c_{12}\alpha_2, \qquad X_2 := \rho\frac{\partial \psi_{in}}{\partial \alpha_2} = \frac{2}{3}c_{12}\alpha_1 + \frac{2}{3}c_{22}\alpha_2 \tag{20}$$

The relations (12), (13), (17) and (18) imply the following remaining inequality

$$(\boldsymbol{\sigma}_1 - \mathbf{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \mathbf{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \mathbf{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) \geq 0 \tag{21}$$

Based on the von Mises stress, we define the quantities

$$J_j := \sigma_{vM}(\boldsymbol{\sigma}_j - \mathbf{X}_j) := \left(\frac{3}{2}(\boldsymbol{\sigma}_j^* - \mathbf{X}_j^*) : (\boldsymbol{\sigma}_j^* - \mathbf{X}_j^*)\right)^{\frac{1}{2}} \qquad j = 1,2 \tag{22}$$

($\sigma_{vM}(\boldsymbol{\sigma})$ - von Mises equivalent stress of $\boldsymbol{\sigma}$) and the *two* yield functions

$$f_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{X}_1, \mathbf{X}_2, R_{0j}) := J_j - R_{0j} \qquad j = 1,2 \tag{23}$$

(R_{0j} is the yield stress of the j^{th} mechanism in case of plasticity. To focus, we do not consider isotropic hardening.) and finally,

$$\mathbf{n}_j := -\frac{\partial f_j}{\partial \mathbf{X}_j} = \frac{3}{2}\frac{\boldsymbol{\sigma}_j^* - \mathbf{X}_j^*}{J_j} \qquad j = 1,2 \tag{24}$$

We assume evolution laws for the inelastic mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ in a *common* form for all inelastic mechanisms:

$$\dot{\boldsymbol{\varepsilon}}_j = \lambda_j \mathbf{n}_j \qquad j = 1,2 \tag{25}$$

The relations (16), (22), (24) and (25) yield

$$\lambda_j = \dot{s}_j \qquad j = 1,2 \tag{26}$$

To distinguish between plastic and creep behaviour, we define the inelastic multipliers λ_j in a suitable way. Clearly, both mechanisms can be of the same kind, but, they can differ, too.

Plastic mechanism: If the j^{th} mechanism is plastic, the (plastic) multiplier $\lambda_j \geq 0$ has to fulfil

$$\lambda_j = 0, \quad if \quad f_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{X}_1, \mathbf{X}_2, R_{0j}) < 0 \tag{27}$$

$$\lambda_j \geq 0, \quad if \quad f_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{X}_1, \mathbf{X}_2, R_{0j}) = 0 \tag{28}$$

As usual in plasticity, λ_j can be expressed via loading conditions (cf. [13], e.g.). In numerical schemes, approximations of λ_j will be determined simultaneously with other quantities.
Creep mechanism: If the j^{th} mechanism models creep, the multiplier λ_j can be defined by

$$\lambda_j := a_j \left(\frac{\sigma_{vM}(\boldsymbol{\sigma}_j - \mathbf{X}_j)}{D_j} \right)^{m_j} s_j^{k_j} \qquad j = 1{,}2 \tag{29}$$

$a_j > 0$, $m_j > 0$, k_j generally depend on temperature θ, s_j is the accumulation in accordance with (16). The drag stress $D_j > 0$ may be constant, or it may have an own evolution (cf. [2], e.g.). Via the exponent k_j the stadia of creep can be distinguished:

- $k_j < 0$ - primary creep,
- $k_j = 0$ - secondary creep,
- $k_j > 0$ - tertiary one.

Clearly, in the case of creep there is no yield stress. Formally, one can use a yield function as in (23) without R_{0j}.

Note that viscoplastic mechanisms can be dealt with analogously. There remain the evolution equations for the internal variables $\boldsymbol{\alpha}_j$. We make a common proposal for plastic and creep behaviour:

$$\dot{\boldsymbol{\alpha}}_j = \dot{\boldsymbol{\varepsilon}}_j - \frac{3}{2} \sum_{j=1}^{2} b_{ji} \mathbf{X}_i \lambda_i \qquad j = 1{,}2 \tag{30}$$

This proposal extends the wide-spread approach which is covered by $b_{12} = b_{21} = 0$ (cf. [2], e.g.). To save thermodynamic consistency, we require that the (generally temperature dependent) matrix **b** is positive semi-definite. However, the matrix **b** is not necessarily symmetric. This gives more possibilities for modelling. We demonstrate this in short. For *constant* c_{ij}, (20) and (30) imply the following generalised Armstrong-Frederick relations (cf. [2], e.g.):

$$\dot{X}_1 = \frac{2}{3} c_{11} \dot{\varepsilon}_1 - c_{11}(b_{11}\mathbf{X}_1 + b_{12}\mathbf{X}_2)\lambda_1 + \frac{2}{3} c_{12} \dot{\varepsilon}_2 - c_{12}(b_{21}\mathbf{X}_1 + b_{22}\mathbf{X}_2)\lambda_2 \tag{31}$$
$$\dot{X}_2 = \frac{2}{3} c_{12} \dot{\varepsilon}_1 - c_{12}(b_{11}\mathbf{X}_1 + b_{12}\mathbf{X}_2)\lambda_1 + \frac{2}{3} c_{22} \dot{\varepsilon}_2 - c_{22}(b_{21}\mathbf{X}_1 + b_{22}\mathbf{X}_2)\lambda_2 \tag{32}$$

If $c_{12} = 0$, $b_{12} = 0$, $b_{11} > 0$, and $b_{22} > 0$, the backstress $\mathbf{X}_1$ has an influence on the evolution of $\mathbf{X}_2$, but not vice versa. Under the assumption "matrix **b** positive semi-definit", the above model is thermodynamically consistent for plastic and creep mechanisms (cf. (21)).

CONCLUSIONS

In this extended abstract, only some basic items of 2M models could be sketched. In our conference contribution, we intend to deal with:

- further approaches for evolution equations,
- thermodynamic consistency in non-standard cases,
- 3M models,
- problems of parameter optimisation,
- formulation of arising mathematical problems.

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