**Proceedings of the 3rd International Conference on Nonlinear Dynamics**
**ND-KhPI2010**
**September 21-24, 2010, Kharkov, Ukraine**

# A CONTACT-STABILIZED NEWMARK METHOD FOR COUPLED DYNAMICAL THERMO-ELASTIC PLOBLEM

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ABSTRACT

A Lagrange multipliers formulation for dynamical frictionless thermo-elastic contact problem is considered. Thermal deformations and dependency of contact thermal resistance on contact pressure are assumed to be the only two coupling effects. Application of standard Newmark method to the problem may lead to spurious numerical oscillations of contact pressures and heat fluxes, inaccurate or divergent solutions. A modification of the Newmark method is proposed where contact contributions are integrated non-monolithically with backward Euler. Elimination of spurious numerical oscillations is shown in a numerical example.

## INTRODUCTION

Dynamical contact problems arise in many practical applications such as turbines, combustion engines and manufacturing. In many cases both mechanical and thermal loads play important role. If contact area and pressure change during the process, then contact heat fluxes vary strongly. The contact heat fluxes influence the temperature distribution and, consequently, thermal deformations, which may cause the change of contact area. Thus, such thermo-mechanical contact problem is intrinsically coupled and non-linear.

One may formulate these contact conditions in a weak form using Lagrange multipliers. Then independent fields of contact pressure and heat fluxes are introduced on the contact interface. Contact pressures play role of Lagrange multipliers for impenetration condition. Heat fluxes satisfy energy balance equations (for details see [1,2]). Spacial descritization of the weak form with FE reduces problem to a system of differential-algebraic equations as follows

$$\begin{cases} \mathbf{M}_{uu}\ddot{\mathbf{d}} + \mathbf{f}_{\text{int}}(\mathbf{d},\dot{\mathbf{d}},\mathbf{T}) + \mathbf{f}_c(\boldsymbol{\lambda},\mathbf{d}) = \mathbf{f}_{ext}(t) \\ \mathbf{M}_{TT}\dot{\mathbf{T}} + \mathbf{r}_{\text{int}}(\dot{\mathbf{d}},\mathbf{T}) + \mathbf{r}_c(\boldsymbol{\lambda},\mathbf{d},\mathbf{T}) = \mathbf{r}_{ext}(t) \\ \mathbf{g}(\mathbf{d}) \geq \mathbf{0},\ \boldsymbol{\lambda} \leq \mathbf{0},\ \mathbf{g}(\mathbf{d})\boldsymbol{\lambda} = \mathbf{0} \\ \mathbf{r}_c(\boldsymbol{\lambda},\mathbf{d},\mathbf{T})\mathbf{g}(\mathbf{d}) = \mathbf{0} \end{cases} \tag{1}$$

with $\mathbf{M}_{uu}$ and $\mathbf{M}_{TT}$ are matrices of mass and heat capacities; $\mathbf{f}_{\text{int}}$ and $\mathbf{r}_{\text{int}}$ are internal force and heat source vectors; vectors $\mathbf{f}_{ext}$ and $\mathbf{r}_{ext}$ are external loads; vectors $\boldsymbol{\lambda},\mathbf{d},\mathbf{T}$ are unknown contact pressures, displacements and temperatures and $\mathbf{f}_c,\mathbf{r}_c$ and $\mathbf{g}$ are contact forces, heat fluxes and constraints.

Initial conditions are specified for displacements, velocities and temperatures. Moreover, initial conditions should not violate contact conditions and be consistent with active constraints, which also imply additional constraints on initial velocities. Initial Lagrange multipliers are recovered from equilibrium [1,3]. Altogether they read as follows

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$$\left\{ \mathbf{d} = \mathbf{d}_0,\ \dot{\mathbf{d}} = \dot{\mathbf{d}}_0,\ \mathbf{T} = \mathbf{T}_0 \right.$$

$$\begin{cases} \mathbf{g}(\mathbf{d}_0) \geq \mathbf{0},\ \mathbf{r}_c(\boldsymbol{\lambda}_0, \mathbf{d}_0, \mathbf{T}_0)\,\mathbf{g}(\mathbf{d}_0) = \mathbf{0} \\ \mathbf{g}(\mathbf{d}_0) \dfrac{d}{d\mathbf{d}} \mathbf{g}\, \dot{\mathbf{d}}_0 = \mathbf{0} \\ \mathbf{g}(\mathbf{d}_0) \left( \dot{\mathbf{d}}_0 \dfrac{d^2}{d\mathbf{d}^2} \mathbf{g}\, \dot{\mathbf{d}}_0 + \dfrac{d}{d\mathbf{d}} \mathbf{g}\, \mathbf{M}_{uu}^{-1} \left[ \mathbf{f}_{ext}(t_0) - \mathbf{f}_{\text{int}}(\mathbf{d}_0, \dot{\mathbf{d}}_0, \mathbf{T}_0) - \mathbf{f}_c(\boldsymbol{\lambda}_0, \mathbf{d}_0) \right] \right) = \mathbf{0} \end{cases} \quad (2)$$

In case of thermo-hyperelastic material, velocities $\dot{\mathbf{d}}$ do not explicitly enter $\mathbf{f}_{\text{int}}$ term. Thus system finally reduces to

$$\begin{cases} \mathbf{M}_{uu}\ddot{\mathbf{d}} + \mathbf{f}_{\text{int}}(\mathbf{d}, \mathbf{T}) + \mathbf{f}_c(\boldsymbol{\lambda}, \mathbf{d}) = \mathbf{f}_{ext}(t) \\ \mathbf{M}_{TT}\ddot{\mathbf{T}} + \mathbf{r}_{\text{int}}(\dot{\mathbf{d}}, \mathbf{T}) + \mathbf{r}_c(\boldsymbol{\lambda}, \mathbf{d}, \mathbf{T}) = \mathbf{r}_{ext}(t) \\ \mathbf{g}(\mathbf{d}) \geq \mathbf{0},\ \boldsymbol{\lambda} \leq \mathbf{0},\ \mathbf{g}(\mathbf{d})\boldsymbol{\lambda} = \mathbf{0} \\ \mathbf{r}_c(\boldsymbol{\lambda}, \mathbf{d}, \mathbf{T})\mathbf{g}(\mathbf{d}) = \mathbf{0} \end{cases} \quad (3)$$

Note, the equations (1.3-5) are nothing else but Karush-Kuhn-Tucker conditions and equation (1.6) means heat insulation in case of positive gap, i.e. contact heat conductance $1/h_c = \Delta T / q_c$ vanishes [4]. Actually, the main challenge arises from non-smooth subsidiary conditions that are illustrated of Fig.1.

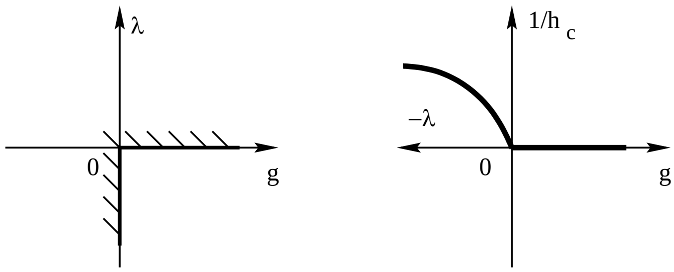



*Fig. 1 Pressure/gap and contact conductance/gap relations*

Rigorous analysis of the systems shows that it is a DAE with differential index 3, i.e. 3 additional differentiations are necessary to transform it into an explicit first-order system (for details see [3]). Such systems are known for number instabilities and numerical problems.

Improper time integration of the system might lead to artificial numerical oscillations of Lagrange multipliers [1-5]. This increases numerical cost and spoils accuracy. In some pathological cases divergent results may be obtained [6].

The most efficient way to repair such defect is to modify an existing time integration scheme with special treatment of the constraints. On the one hand ordinary users are familiar with such methods. On the other hand only few coding is necessary to get valuable results. Newmark method gives such opportunity. As backward methods are generally known for their stability, one can modify predictor to treat contact constraints using backward Euler (due to ideas of Lane et. al. [7]). The other idea is to include an additional projector on the predictor step. Standard Newmark predictor leads to strong violation of the constraints, which means expensive correction phase [5]. Both methods are reported to be successful in elimination of artificial oscillations for mechanical problems. However, they introduce artificial damping and generally are not energy preserving [5,7].

Here we present modification of the predictor step with backward Euler integration of the contact contributions that is extended for thermo-elastic problem. A numerical example illustrates efficiency of the proposed approach.

## 1. Standard Newmark scheme for thermo-elastic contact problem

As a starting point we use standard Newmark method (see [2,5,7]). It assumes following integration rule for the variables, predictor and corrector

$$\begin{cases} \mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t\, \dot{\mathbf{d}}_n + \dfrac{\Delta t^2}{2}\left((1-2\beta)\ddot{\mathbf{d}}_n + 2\beta\ddot{\mathbf{d}}_{n+1}\right) \\ \dot{\mathbf{d}}_{n+1} = \dot{\mathbf{d}}_n + \Delta t\left((1-\gamma)\ddot{\mathbf{d}}_n + \gamma\ddot{\mathbf{d}}_{n+1}\right) \\ \mathbf{T}_{n+1} = \mathbf{T}_n + \Delta t\left((1-\gamma)\dot{\mathbf{T}}_n + \gamma\,\dot{\mathbf{T}}_{n+1}\right) \end{cases} \tag{4}$$

$$\begin{cases} \tilde{\mathbf{d}}_{n+1} = \mathbf{d}_n + \Delta t\, \dot{\mathbf{d}}_n + \dfrac{\Delta t^2}{2}(1-2\beta)\mathbf{M}_{uu}^{-1}\left(\mathbf{f}_{ext,n} - \mathbf{f}_{\mathrm{int},n} - \mathbf{f}_{c,n}\right) \\ \tilde{\mathbf{T}}_{n+1} = \mathbf{T}_n + \Delta t(1-\gamma)\mathbf{M}_{TT}^{-1}\left(\mathbf{r}_{ext,n} - \mathbf{r}_{\mathrm{int},n} - \mathbf{r}_{c,n}\right) \end{cases} \tag{5}$$

$$\begin{cases} \mathbf{d}_{n+1} = \tilde{\mathbf{d}}_{n+1} + \beta\Delta t^2\mathbf{M}_{uu}^{-1}\left(\mathbf{f}_{ext,n+1} - \mathbf{f}_{\mathrm{int},n+1} - \mathbf{f}_{c,n+1}\right) \\ \mathbf{T}_{n+1} = \tilde{\mathbf{T}}_{n+1} + \Delta t\gamma\,\mathbf{M}_{TT}^{-1}\left(\mathbf{r}_{ext,n+1} - \mathbf{r}_{\mathrm{int},n+1} - \mathbf{r}_{c,n+1}\right) \\ \mathbf{g}(\mathbf{d}_{n+1}) \ge \mathbf{0},\ \boldsymbol{\lambda}_{n+1} \le \mathbf{0},\ \mathbf{g}(\mathbf{d}_{n+1})\boldsymbol{\lambda}_{n+1} = \mathbf{0} \\ \mathbf{r}_c(\boldsymbol{\lambda}_{n+1}, \mathbf{d}_{n+1}, \mathbf{T}_{n+1})\mathbf{g}(\mathbf{d}_{n+1}) = \mathbf{0} \end{cases} \tag{6}$$

The corrector system of equations (6) is implicit, i.e. $\mathbf{d}_{n+1}, \mathbf{T}_{n+1}$ and $\boldsymbol{\lambda}_{n+1}$ enters both left and right hand side of equation. It is also a non-linear system, which means it should be solved iteratively, i.e. with Newton-Raphson method. A consistent with (6) tangent reads as follows

$$\mathbf{A} = \begin{bmatrix} \dfrac{1}{\beta\Delta t^2}\mathbf{M}_{uu} + \mathbf{K}_{uu} + \boldsymbol{\lambda}\mathbf{H} & \mathbf{K}_{uT} & \mathbf{G}^T \\ \mathbf{D}_{Tu} + \mathbf{K}_{Tu}^c & \dfrac{1}{\gamma\Delta t}\mathbf{M}_{TT} + \mathbf{K}_{TT} + \mathbf{K}_{TT}^c & \mathbf{K}_{T\lambda} \\ \mathbf{G} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{7}$$

with $\mathbf{G} = \dfrac{d\mathbf{g}}{d\mathbf{d}}$ and $\mathbf{H} = \dfrac{d^2\mathbf{g}}{d\mathbf{d}^2}$ − gradient and Hessian of constraints; $\mathbf{K}_{uu} = \dfrac{d\mathbf{f}_{\mathrm{int}}}{d\mathbf{d}}$ and $\mathbf{K}_{uT} = \dfrac{d\mathbf{f}_{\mathrm{int}}}{d\mathbf{T}}$ − tangent stiffness and thermoelastic stiffness; $\mathbf{K}_{TT} = \dfrac{d\mathbf{r}_{\mathrm{int}}}{d\mathbf{T}}$, $\mathbf{K}_{Tu}^c = \dfrac{d\mathbf{r}_c}{d\mathbf{d}}$, $\mathbf{K}_{T\lambda} = \dfrac{d\mathbf{r}_c}{d\boldsymbol{\lambda}}$ and $\mathbf{K}_{TT}^c = \dfrac{d\mathbf{r}_c}{d\mathbf{T}}$ − conductivity matrices; $\mathbf{D}_{Tu}$ − thermoelastic damping matrix.

## 2. Modified predictor for thermo-elastic contact problem

Instead of applying monolithic integration rule for external, internal, inertial and contact forces, it is suggested to integrate $\mathbf{f}_c$ and $\mathbf{r}_c$ non-monolithically with backward Euler. It doesn't change update rule (4), because contact terms do not explicitly enter it. The standard predictor (5) uses contact contributions on the previous step $\mathbf{f}_{c,n}$ and $\mathbf{r}_{c,n}$. Now we exclude them from the predictor. Contribution of new values $\mathbf{f}_{c,n+1}$ and $\mathbf{r}_{c,n+1}$ in corrector is calculated as $\Delta t^2\mathbf{M}_{uu}^{-1}\mathbf{f}_{c,n+1}$ and $\Delta t\,\mathbf{M}_{TT}^{-1}\mathbf{r}_{c,n+1}$. Thus consistent expressions for predictor, corrector and algorithmic tangent read

$$\begin{cases} \tilde{\mathbf{d}}_{n+1} = \mathbf{d}_n + \Delta t\, \dot{\mathbf{d}}_n + \dfrac{\Delta t^2}{2}(1-2\beta)\mathbf{M}_{uu}^{-1}\left(\mathbf{f}_{ext,n} - \mathbf{f}_{\text{int},n}\right) \\ \tilde{\mathbf{T}}_{n+1} = \mathbf{T}_n + \Delta t(1-\gamma)\mathbf{M}_{TT}^{-1}\left(\mathbf{r}_{ext,n} - \mathbf{r}_{\text{int},n}\right) \end{cases} \tag{8}$$

$$\begin{cases} \mathbf{d}_{n+1} = \tilde{\mathbf{d}}_{n+1} + \beta\Delta t^2 \mathbf{M}_{uu}^{-1}\left(\mathbf{f}_{ext,n+1} - \mathbf{f}_{\text{int},n+1} - \mathbf{f}_{c,n+1}/\beta\right) \\ \mathbf{T}_{n+1} = \tilde{\mathbf{T}}_{n+1} + \Delta t\gamma\, \mathbf{M}_{TT}^{-1}\left(\mathbf{r}_{ext,n+1} - \mathbf{r}_{\text{int},n+1} - \mathbf{r}_{c,n+1}/\gamma\right) \\ \mathbf{g}(\mathbf{d}_{n+1}) \geq \mathbf{0},\ \boldsymbol{\lambda}_{n+1} \leq \mathbf{0},\ \mathbf{g}(\mathbf{d}_{n+1})\boldsymbol{\lambda}_{n+1} = \mathbf{0} \\ \mathbf{r}_c(\boldsymbol{\lambda}_{n+1}, \mathbf{d}_{n+1}, \mathbf{T}_{n+1})\mathbf{g}(\mathbf{d}_{n+1}) = \mathbf{0} \end{cases} \tag{9}$$

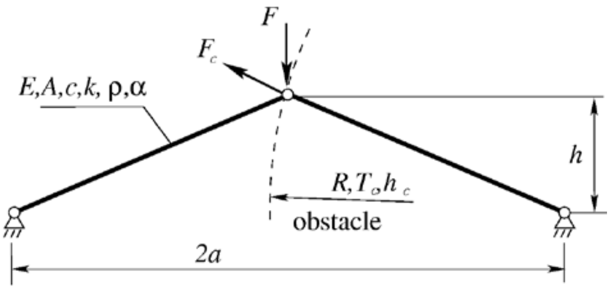
$$\mathbf{A} = \begin{bmatrix} \dfrac{1}{\beta\Delta t^2}\mathbf{M}_{uu} + \mathbf{K}_{uu} + \boldsymbol{\lambda}\mathbf{H}/\beta & \mathbf{K}_{uT} & \mathbf{G}^T/\beta \\ \mathbf{D}_{Tu} + \mathbf{K}_{Tu}^c/\gamma & \dfrac{1}{\gamma\Delta t}\mathbf{M}_{TT} + \mathbf{K}_{TT} + \mathbf{K}_{TT}^c/\gamma & \mathbf{K}_{T\lambda}/\gamma \\ \mathbf{G} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{10}$$

The advantages of the proposed modification are straightforward implementation and consistent coupled time integration. The disadvantages are two additional matrix inversions in predictor step, zeroes on diagonal of algorithmic tangent and lack of its symmetry. If we use lumped matrixes the overhead of matrix inversions is neglectable [5]. Usage of dual Lagrange multipliers allows us to eliminate zeroes on diagonal [1]. But, unfortunately, symmetric algorithmic tangent cannot be achieved within proposed approach.

## 3. Numerical example

Proposed algorithm was initially implemented and tested in computer algebra system Maple. As a numerical example we chose a problem of dynamical snap-through of a shallow arch. Despite simplicity of the example, it shows spurious oscillation of contact resultants, large sliding contact with high degree of nonlinearity and sufficient coupling between fields.

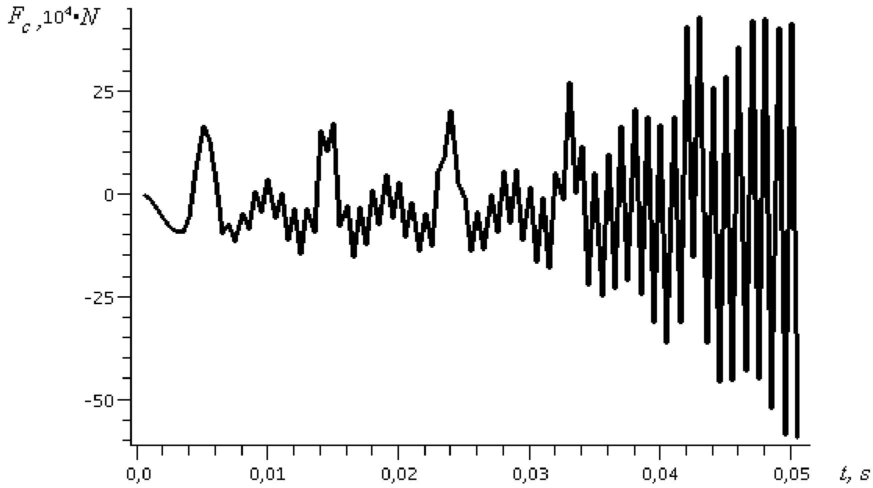
Two thermo-hyperelastic truss elements are used (St. Venant-Kirchhoff material [1]). Abrupt force $F$ is applied in vertical direction. In addition the middle node is constrained to slide along rigid circle as shown on Fig.2. The temperature of the obstacle was defined as function of vertical displacement $T_c = (372 - 100\, d_y)$ K, which makes term $\mathbf{K}_{Tu}^c \neq \mathbf{0}$.




$E = 2.1 \cdot 10^{11}$ Pa  
$\mathrm{A} = 10^{-4}\ \mathrm{m}^2$  
$\rho = 7850\,\mathrm{kg/m}^3$  
$\alpha = 1.25 \cdot 10^{-5}\ \mathrm{K}^{-1}$  
$k = 52\ \mathrm{W/m \cdot K}$  
$c = 0.5\,\mathrm{J/kg \cdot K}$  
$a = 1$ m  
$h = 0.1$ m  
$h_c = \left(0.1 + 10^{-9}\lambda^2\right)\mathrm{W/K}$  
$R = 2$ m

*Fig.2 Setup of numerical example*

Both standard and modified schemes were tested with default parameters for Newmark $\beta = 0.25$, $\gamma = 0.5$, and constant time step $\Delta t = 0.0005\,\mathbf{s}$. Lagrange multipliers over time are shown on Fig.3.



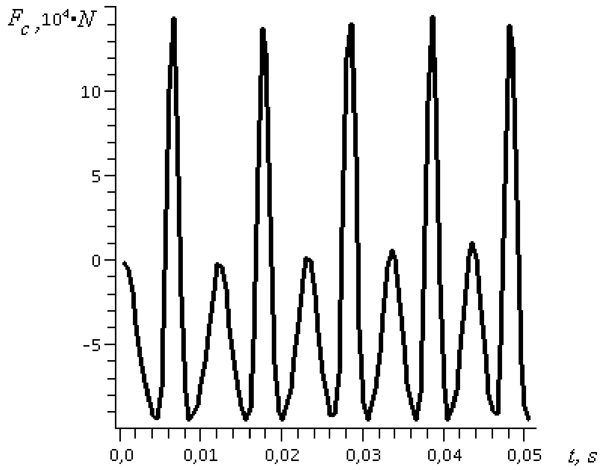




*Fig.3. Lagrange multipliers from standard (upper) vs. modified (below) scheme*

More pronounce difference shows up in temperature at middle node. Overestimation of contact force leads to overestimation of contact heat conductance and contact heat fluxes. Therefore the standard Newmark scheme fails to predict correctly temperatures (Fig.4) and should not be used for this problem together with Lagrange multipliers formulation (however, we did not study behavior of standard scheme together with penalty formulation).

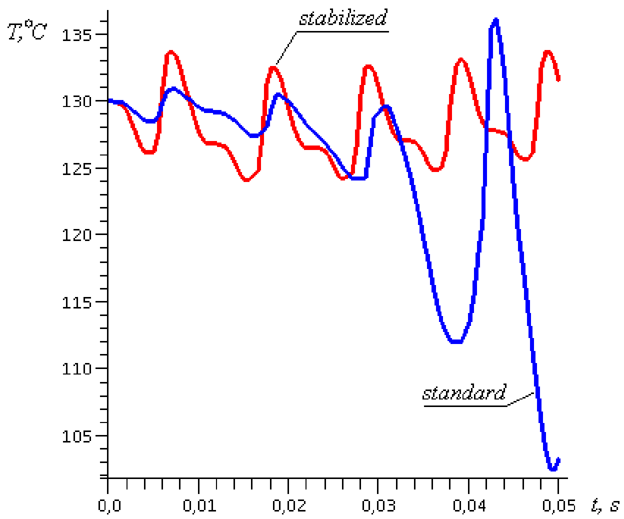



*Fig.4. Temperature at middle node (°C) over time*

## CONCLUSIONS

Standard Newmark scheme may fail for dynamical thermo-elastic contact problem. Modification of predictor/corrector of Newmark method is proposed. It is shown that this modification eliminates oscillation of Lagrange multipliers. In the future we plan to implement the method to two-body frictional contact in 3D, study the question whether it is necessary to do an additional projection to admissible set during predictor step.

## ACKNOWLEDGMENTS

This work couldn't be done without continuous discussions and encouragement from my colleagues Thomas Cichosz and Manfred Bischoff from Institute für Baustatik und Baudynamik, Stuttgart University, Germany.

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