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A HYBRID ASYMPTOTIC WKB-GALERKIN METHOD WITH APPLICATION TO THE CORRELATION ANALYSIS OF STOCHASTIC BEHAVIOUR OF NON-LINEAR SYSTEMS WITH TIME-DEPENDED PARAMETERS

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ABSTRACT

The method of double asymptotic expansion in aggregate with a hybrid WKB method on the base of the energy conservation law is examined in the frame of the correlation analysis of stochastic behavior of nonlinear system with time-depended parameters under casual loading. The application of the asymptotic approaches for the analysis of forced oscillations systems with variable factor of damping on the basis of the stochastic nonlinear Duffing's equation is done. The solution is searched as a series on degrees of small parameter at nonlinear component of initial equation (external asimptotic). In this case, the initial equation can be replaced by a recurrent sequence of linear equations. The solution of the given system can be found by a method of Green's functions and WKB method (internal asymptotic). The energy conservation law is used for improving solution. Moment functions of output process are defined by average of a series. The results of visualization of determenistic solution and correlation function of output process are given.

INTRODUCTION

In the paper a hybrid technique [1, 2] for obtaining of the approximate analytical solution of the second order nonlinear differential equations of type (1) with time-dependent parameters and initial conditions is applied. Generally, hybrid technique is based on using classical perturbation methods combined with some principles of definition of artificial unknown coefficients in these expansions [3], [4]. For example the Galerkin's orthogonality and variational principles, the method of least squares etc.

However, some of the above mentioned principles can be applied successfully to the solution of time-dependent problems. For example, variational principles used in the Euler's equation do not work because for time-dependent problems only the initial conditions at some moment of time are known. As well the Galerkin's orthogonality procedure does not give the reasonable results in problems for equations with variable coefficients. In this paper as the principles of definition of artificial unknown coefficients at functions of asymptotic expansions the Hamilton's principle combined with the method of the least squares are used. Approximate asymptotic solution of nonlinear equation is found with the help of the method of double asymptotic expansion [5]. Then the solution is twice specified with the help of the described hybrid technique which is based on the WKBJ-Galerkin method (internal asymptotic) [3] and the perturbation-Galerkin method (external asymptotic) [4]. On the basis of the analytical solution partial expressions for the correlation function of the output process under random loading are obtained.

1. DESCRIPTION OF THE HYBRID TECHNIQUE

We consider a nonlinear differential equation of the second order:

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$$L_0 f(t) + \alpha \cdot \left(P(t) f^2(t) + Q(t) f^3(t)\right) = \gamma(t), \quad (1)$$

where L_0 is a linear differential operator:

$$L_0 = \frac{\partial^2}{\partial t^2} + 2c(t)\frac{\partial}{\partial t} + \omega_0^2 \varphi(t); \quad (2)$$

t is time variable; $c(t)$, $\varphi(t)$, $P(t)$, $Q(t)$ are some functions of time, which depend on characteristics of plate and function of the external loading; $\gamma(t)$ is a function of the external loading; ω_0, α are parameters of frequency of natural vibrations of the linear system and degree of non-linearity.

We will find the solution of the equation (1) in the interval of time. On the first stage of the approach we obtain the solution of equation (1) in the form of a series:

$$f(t) = f_{\text{hom}}(t) + \sum_{j=0}^{\infty} \alpha^j f_j(t), \quad (3)$$

where: $f_{\text{hom}}(t)$, $f_j(t)$ (j = 0, 1, ...) are unknown functions of time.

It is necessary to substitute an expression (3) and its first two derivatives into equation (1) and splitting it with powers of parameter α one may obtain the recurrent system of linear differential equations as for $f_j(t)$ (j = 0, 1 ...)

$$L_0 f_{\text{hom}}(t) = 0 \quad (4)$$

$$\begin{aligned} &L_0 f_0(t) = \gamma(t), \\ &L_0 f_1(t) = -\left(P(t)\left(f_{\text{hom}}(t) + f_0(t)\right)^2 + Q(t)\left(f_{\text{hom}}(t) + f_0(t)\right)^3\right), \\ &\ldots \end{aligned} \quad (5)$$

Let the operator reverse to L_0 corresponds to the Volter's operator with the Green's function which is a solution of differential equation

$$L_0 h(t,\tau) = 0. \quad (6)$$

Then on the second stage it is possible to find the solution of equations of the system (5) with the help of Green's functions method [6]:

$$\begin{aligned} &f_0(t) = \int_{t_0}^{t} h(t,\tau)\gamma(\tau)\, d\tau, \\ &f_1(t) = -\int_{t_0}^{t} h(t,\tau)\left(P(t)\left(f_{\text{hom}}(t) + f_0(t)\right)^2 + Q(t)\left(f_{\text{hom}}(t) + f_0(t)\right)^3\right) d\tau, \\ &\ldots \end{aligned} \quad (7)$$

On the third stage, we will find the solutions of linear homogeneous differential equations (4) and (6) with the help of the WKBJ-method in the form:

$$f_{\text{hom}}(t) = \exp\left(\int_{t_0}^{t} \sum_{k=0}^{\infty} \varepsilon^{k-1} \psi_k(\theta)\, d\theta\right), \quad h(t,\tau) = \exp\left(\int_{\tau}^{t} \sum_{k=0}^{\infty} \varepsilon^{k-1} \psi_k(\theta)\, d\theta\right) \quad (8)$$

where: $\varepsilon = 1/\omega_0$; $\psi_k(t)$ are unknown functions of time (k = 0, 1, ...).

Substituting expressions (8) into equations (4) and (6) and collecting coefficients at the degrees of parameter ε, we get the system of equations for functions $\psi_k(t)$ $(k = 0, 1 \dots)$ and then

$$\psi_{0_{1,2}}(t) = \pm i\sqrt{\varphi_c(t)},$$
$$\psi_{1_{1,2}}(t) = -\frac{\psi'_0(t)}{2\psi_0(t)} = -\frac{\varphi'_c(t)}{4\varphi_c(t)} = -\frac{1}{4}(\ln\varphi_c(t))', \quad (9)$$
$$\dots,$$

where

$$\varphi_c(t) = \varphi(t) - \frac{c^2(t)}{\omega_0^2} - \frac{c'(t)}{\omega_0^2}. \quad (10)$$

On the fourth stage according to WKBJ-Galerkin method [3], the hybrid solutions of equations (4) and (6) can be represented in the form as follows

$$f_{\hom_H}(t) = \exp\left(\int_{t_0}^{t}\sum_{k=0}^{M-1}\lambda_k\psi_k(\theta)\,d\theta\right), \quad h_H(t,\tau) = \exp\left(\int_{\tau}^{t}\sum_{k=0}^{M-1}\mu_k\psi_k(\theta)\,d\theta\right), \quad (11)$$

where: M is an order of approaching; $\psi_k(t)$ $(k = 0, \dots, M-1)$ are functions of time, determined on the third stage, λ_k, μ_k $(k = 0, \dots, M-1)$ are unknown coefficients which depend on the parameter ε.

Finally, if the Green's function is known, on the fifth stage by the method of perturbation-Galerkin [4], hybrid solution $f_H(t)$ of initial differential equation (1) can be represented as [6]

$$f_H(t) = f_{\hom_H}(t) + \sum_{j=0}^{N-1}\delta_j f_{j_H}(t), \quad (12)$$

where: N is an order of approaching; δ_j $(j = 1, \dots, N-1)$ are unknown coefficients which depend on the parameters ε and α;

$$f_{0_H}(t) = \int_{t_0}^{t} h_H(t,\tau)\gamma(\tau)\,d\tau$$
$$f_{1_H}(t) = -\int_{t_0}^{t} h_H(t,\tau)\Big(P(t)\big(f_{\hom_H}(t) + f_{0_H}(t)\big)^2 + Q(t)\big(f_{\hom_H}(t) + f_{0_H}(t)\big)^3\Big)d\tau, \quad (13)$$
$$\dots$$

We may determine unknown coefficients λ_k, μ_k $(k = 0, \dots, M-1)$ and δ_j $(j = 1, \dots, N-1)$ with help of the energy conservation law:

$$E(t) - E(t_0) = W(t), \quad (14)$$

where E is the complete energy of the system:

$$E(t) = T(t) + U(t); \quad (15)$$

W is work of external and internal forces; U is potential energy; T is kinetic energy.

In our case of nonlinear vibrations of a plate we may rewrite the expression (14) in the form:

$$\int_{t_0}^{t} f'(\tau)\cdot\left(L_0 f(\tau)+\alpha\cdot\left(P(\tau)f^2(\tau)+Q(\tau)f^3(\tau)\right)-\gamma(\tau)\right)d\tau=0 \qquad (16)$$

In general case eq. (16) is not performed. Therefore for determination of unknown coefficients a least-squares method is applicable:

$$\int_{t_0}^{T}\left(E(t)-E(t_0)-W(t)\right)^2 dt \to \min . \qquad (17)$$

2. KORRELATION FUNCTIONS

The moment functions of output process (functions of bending of plate) are determined by averaging of a series (12). Let the external load $\gamma(t)$ is a centralized random process, then the output process will be centralized too. We suppose also, that initial conditions are zero. Thus, for the second-order moment function we obtain:

$$\left\langle f_H(t_1) f_H(t_2)\right\rangle=\delta_0^2\left\langle f_{0_H}(t_1) f_{0_H}(t_2)\right\rangle+\delta_0\delta_1\left(\left\langle f_{0_H}(t_1) f_{1_H}(t_2)\right\rangle+\left\langle f_{0_H}(t_2) f_{1_H}(t_1)\right\rangle\right)+\ldots, \qquad (18)$$

where

$$\begin{aligned}\left\langle f_{0_H}(t_1) f_{1_H}(t_2)\right\rangle&=-\delta_0\delta_1\int_{t_0}^{t_2} h_H(t_2,\tau)\\ &\left(P(\tau)\left\langle f_{0_H}(t_1) f^2_{0_H}(\tau)\right\rangle+Q(\tau)\left\langle f_{0_H}(t_1) f^3_{0_H}(\tau)\right\rangle\right)d\tau\end{aligned} \qquad (19)$$

Here and below the angular brackets denote the mathematical expectation.

Let external load $\gamma(t)$ is a normal random process. Then the process $f_{0_H}(t_1)$ will be the normal random process as well. In this case the moment functions of the odd order of the zero-order approximation $f_0(t)$ are equal to the zero. There certain relationships exist [6]:

$$\begin{aligned}\left\langle f_{0_H}(t_1) f^3_{0_H}(t_2)\right\rangle&=3K_{f_0}(t_1,t_2)K_{f_0}(t_2,t_2)\\ \left\langle f_{0_H}(t_1) f^2_{0_H}(t_2)\right\rangle&=0\end{aligned}, \qquad (20)$$

where $K_{f_0}(t_1,t_2)$ is the correlation function of random process $f_{0_H}(t)$.

Substituting obtained results into expression (19), taking in account that both input and output processes are centralised, we get the final expression for the correlation function of the output process

$$K_f(t_1,t_2)\approx\delta_0^2 K_{f_0}(t_1,t_2)+\delta_0\delta_1\left(K_{f_0 f_1}(t_1,t_2)+K_{f_0 f_1}(t_2,t_1)\right) \qquad (21)$$

where $K_f(t_1,t_2)$ is the correlation function of the output process,

$$K_{f_0 f_1}(t_1,t_2)=-3\int_{t_0}^{t_2} h_H(t_2,\tau)\,P(\tau)K_{f_0}(t_1,\tau)K_{f_0}(\tau,\tau)d\tau, \qquad (22)$$

$K_{f_0}(t_1,t_2)$ is the correlation function of the zero-order approximation:

$$K_{f_0}(t_1,t_2)=\int_{t_0}^{t_1}\int_{t_0}^{t_2} h_H(t_1,\tau_1)h_H(t_2,\tau_2)K_\gamma(\tau_1,\tau_2)d\tau_1 d\tau_2, \qquad (23)$$

$K_\gamma(t_1, t_2)$ is the correlation function of external load.

Terms of series (12) with squares and highest orders of parameter α, will depend on moment functions of the processes $f_1(t)$, $f_2(t)$, … Under normal external loading given processes will not be normal at all. Therefore, computation of next terms of expansion will cause difficulties. To overcome these difficulties we have to introduce additional hypotheses on moment functions.

1. NUMERICAL RESULTS

We plot asymptotic (on the basis of the WKBJ method and perturbation method at $M = 2$ and $N = 2$), hybrid (at $M = 2$ and $N = 2$), numerical and linear solutions of equation (1). Numerical realization is represented for the following parameters of equation: $t_0 = 0$, $T = 40$ (i.e. $t \in [0,40]$), $\varphi = t/T + 1/10$, $\gamma = 1/(2T)$, $\omega_0 = 1$.

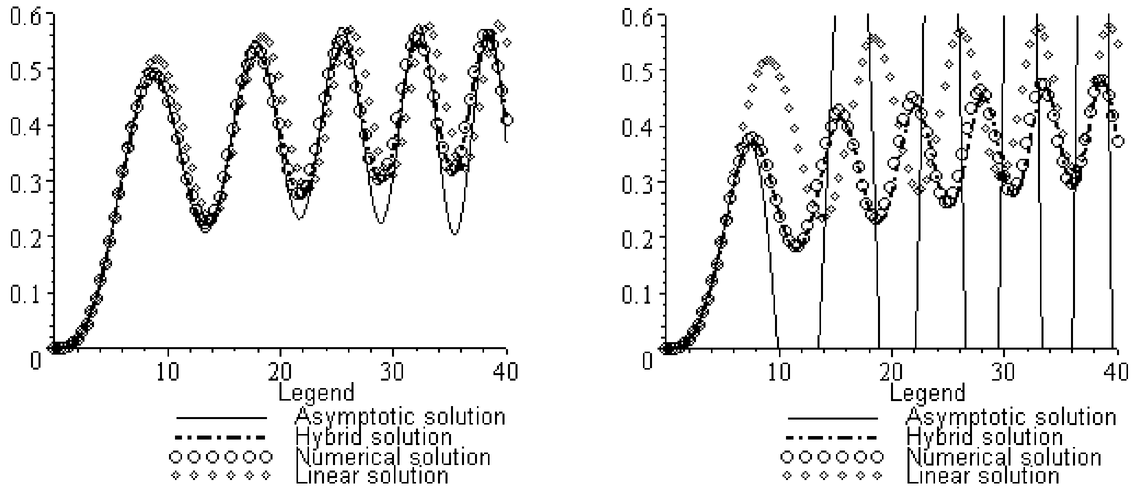

Fig. 1 Comparison of hybrid (dash-dot line) solution of the equation (1) with asymptotic (solid line), numerical (circles) and linear (diamonds) solutions;
$P = 0$; $Q = 1$; $f(0) = 0$; $f'(0) = 0$; *a)* $\alpha = 0,1$; *b)* $\alpha = 1$.

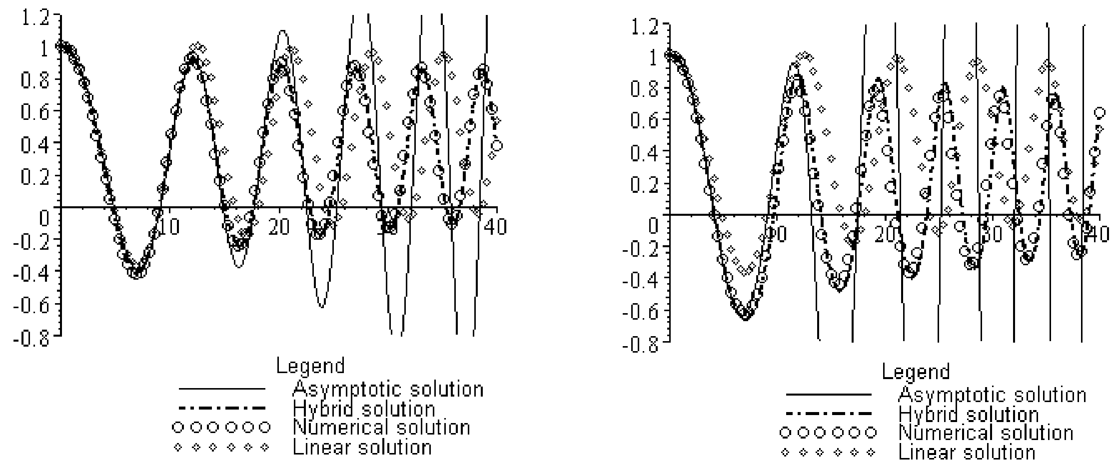

Fig. 2 Comparison of hybrid (dash-dot line) solution of the equation (1) with asymptotic (solid line), numerical (circles) and linear (diamonds) solutions;
$P = 3Q$; $Q = t/T$; $f(0) = 1$; $f'(0) = 0$; *a)* $\alpha = 0,1$; *b)* $\alpha = 0,5$.

As it is shown solutions obtained by the hybrid approach compared well with numerical results on more wide ranges of change of parameter of non-linearity.

On the fig. 3-4 the results of visualization of dispersion $D_f(t) = K_f(t,t)$ and the correlation function $K_f(t,t)$ of output process $f(t)$ of the nonlinear system (1) are presented. For all graphs

$t_0 = 0$; $T = 3 \cdot 2\pi / \omega_0 = 3 \cdot 2\pi\varepsilon$, where $2\pi / \omega_0 = 2\pi\varepsilon$ – period of vibrations of a similar linear system; $\varphi(t) = 1 - \frac{(t - t_0)}{T - t_0} \frac{w_0^2}{10}$ (i.e. a function $\varphi(t)$ varies linearly from one to $1 - \omega_0^2 / 10$).

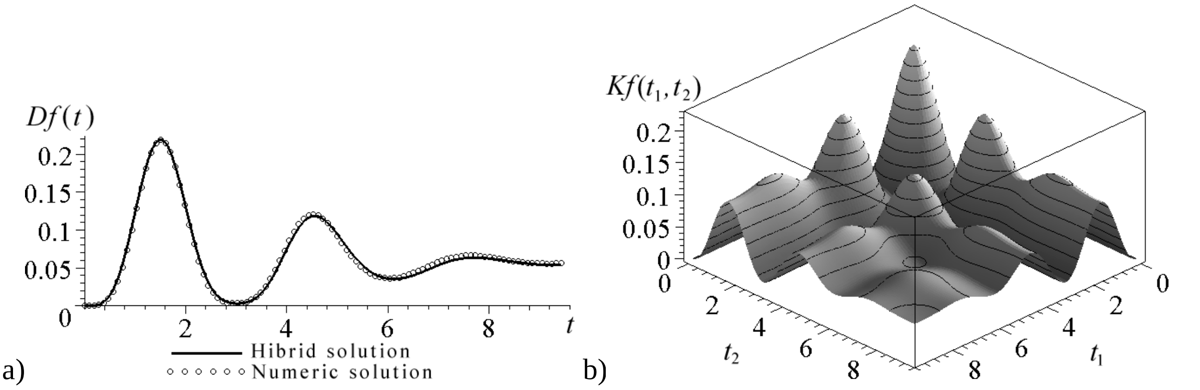

Fig. 3 The graphics of dispersion and correlation function of output process; $\omega_0 = 2$; $\alpha = 2$; $K_\gamma(t_1, t_2) = 1$; a) dispersion; b) correlation function.

On the fig. 4 the results of visualization of dispersion $D_f(t) = K_f(t,t)$ and correlation function $K_f(t,t)$ of output process $f(t)$ of the nonlinear system under loading of «white noise» type are presented.

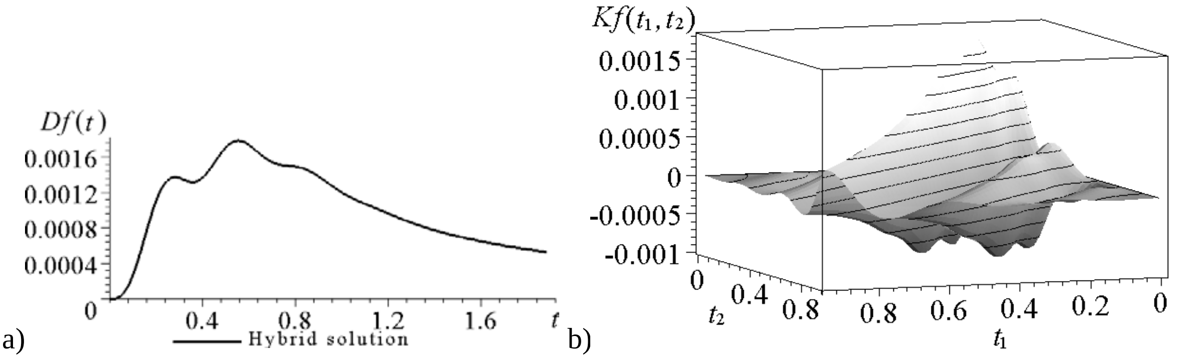

Fig. 4 The graphics of dispersion and correlation function of output process; $\omega_0 = 10$; $\alpha = 10$; $K_\gamma(t_1, t_2) = \delta(t_2 - t_1)$; a) dispersion; b) correlation function.

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