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# ANALYSIS OF NONLINEAR ANHARMONIC PERTURBATIONS FOR AXISYMMETRIC LONGITUDINAL-SHEAR WAVES IN A CYLINDRICAL WAVEGUIDE

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ABSTRACT

The numerical-analytical solution of the nonlinear generation of second harmonic axisymmetric normal elastic waves of longitudinal-shear type in a circular aluminum waveguide is obtained. Applied model of geometrically and physically nonlinear deformation is based on the Murnaghan elastic potential, a representation of finite deformations and method of decomposition of nonlinear wave movements in the rows of the small parameter.

## INTRODUCTION

Investigation of nonlinear effects in the propagation of normal elastic waves in deformable waveguides of different geometric shapes has a number of important engineering applications [1,2]. It gives information about the interaction of elastic waves, which is described in the linear approximation as independent and allows describe the anharmonic effects of appearance in the waveguides waves with double frequencies. The fields of application of research of nonlinear anharmonic effects are ultrasonic wave diagnosis, seismic and acoustic electronics. Nonlinear properties of elastic waves are used in the concept of acoustoelectronic devices for integration of signals, ultrasonic convolver.

Several problems of this type are uninvestigated because of the extreme complexity of the theoretical solutions.. These include the current problem of anharmonic effects in the propagation of normal elastic waves in cylindrical three-dimensional geometry waveguides. Some aspects of the problem for the cylindrical waveguide were considered in [3,4].

## 1. FORMULATION AND METHOD OF SOLUTION

In this paper the numerical-analytical problem of definition and investigation of nonlinear second harmonics monochromatic axisymmetric normal longitudinal waves propagating along the axial direction in an isotropic circular section cylinder is presented. The waveguide in normalized cylindrical coordinates occupies the region

$$V = \{0 \le r \le R, 0 \le \theta \le 2\pi, -\infty < z < \infty\} \tag{1}$$

The lateral surface of the cylinder is rigidly fixed. Characteristics of the investigated wave field are complex functions of the wave elastic displacements $u_\alpha(r,z,t)$ $(\alpha = r,z)$.

The model of nonlinear dynamic deformation of isotropic elastic media was used, taking into account the effects of geometrical and physical nonlinearity. It includes the representation of the elastic Murnaghan potential $U$ with quadratic and cubic terms of finite deformation $E_{\alpha\beta}$ $(\alpha,\beta,\gamma = r,\theta,z)$

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$$U = \frac{\lambda + 2\mu}{2} E_1^2 - 2\mu E_2 + \frac{l + 2m}{3} E_1^3 - 2mE_1E_2 + nE_3 ; \quad (2)$$

where

$$E_1 = I_1 , \; E_2 = \frac{1}{2}(I_1^2 - I_2) , \; E_3 = \frac{1}{6}(I_1^3 - 3I_1I_2 + 2I_3) ;$$

$I_j$ – invariants of the strain tensor

$$I_1 = E_{rr} + E_{\theta\theta} + E_{zz} , \quad I_2 = E_{\theta\theta}E_{zz} + E_{\theta\theta}E_{rr} + E_{zz}E_{rr} - E_{rz}E_{zr} , \quad I_3 = E_{rr}E_{\theta\theta}E_{zz} - E_{rz}E_{\theta\theta}E_{zr} ,$$

$\lambda, \mu$ - Lame parameters; $l, m, n$ - elastic constants of second order for the material of the cylinder. Nonlinear representation of the tensor components of elastic deformations $E_{\alpha\beta}$ in axisymmetrical case are given by

$$E_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}((\frac{\partial u_r}{\partial r})^2 + (\frac{\partial u_z}{\partial r})^2), \quad E_{\theta\theta} = \frac{u_r}{r} + (\frac{u_r}{r})^2, \quad E_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}((\frac{\partial u_r}{\partial z})^2 + (\frac{\partial u_z}{\partial z})^2), \quad (3)$$

$$E_{\theta z} \equiv 0 , \quad E_{rz} = \frac{1}{2}(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}), \quad E_{r\theta} \equiv 0 .$$

Representation of the second component of the Piola-Kirchhoff stress tensor at the base venues of the cylindrical coordinate system is consistent with this form of elastic potential and has the form

$$T_{\alpha\beta} = \frac{2\sigma}{1 - 2\sigma} I_1 \delta_{\alpha\beta} + 2E_{\alpha\beta} + [\frac{l}{\mu} I_1^2 - (2\frac{m}{\mu} - \frac{n}{\mu}) I_2 ]\delta_{\alpha\beta} + (2\frac{m}{\mu} - \frac{n}{\mu}) I_1 E_{\alpha\beta} + \frac{n}{\mu} E_{\alpha\gamma} E_{\gamma\beta}$$

$$(\alpha, \beta, \gamma = r, \theta, z) , \quad (4)$$

where $\delta_{\alpha\beta}$– components of the unit tensor, $\sigma$- Poisson's ratio. The Lagrange stress tensor components on the main venues of the cylindrical coordinate system in axisymmetrical case, respectively, are determined by the relations

$$\begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\partial u_r}{\partial r} & 0 & \frac{\partial u_r}{\partial z} \\ 0 & 1 + \frac{u_r}{r} & 0 \\ \frac{\partial u_z}{\partial r} & 0 & 1 + \frac{\partial u_z}{\partial z} \end{bmatrix} \begin{bmatrix} T_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & T_{zz} \end{bmatrix} . \quad (5)$$

In the differential form the boundary value problem, describes the investigated nonlinear wave field, involves the equation of the dynamic deformation

$$\frac{1}{r}\frac{\partial}{\partial r}(rS_{rr}) + \frac{\partial S_{rz}}{\partial z} - \frac{S_{\theta\theta}}{r} + \rho\frac{\partial^2 u_r}{\partial t^2} = 0 , \quad \frac{1}{r}\frac{\partial}{\partial r}(rS_{zr}) + \frac{\partial S_{zz}}{\partial z} + \rho\frac{\partial^2 u_z}{\partial t^2} = 0 \quad (6)$$

and boundary conditions, which have the following form in the case of rigidly fixed lateral surface

$$(u_r)_{r=R} = (u_z)_{r=R} = 0 ;$$

In the equations (6) $\rho$ is the density of the material of the cylinder in the undeformed state.

For the analysis of small nonlinear perturbations (anharmonic effects) in this paper it is used the search method of successive approximations, described by the first linear and second nonlinear harmonics of normal elastic waves. Analysis of nonlinear effects in the propagation of normal axisymmetric longitudinal waves for a cylinder with a rigidly clamped lateral surface in the first linear approximation reduces to the next homogeneous spectral boundary value problem for the functions of the wave elastic displacements $u_r^{(l)}, u_z^{(l)}$ :

$$\rho \frac{\partial^2 u_r^{(l)}}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}(r S_{rr}^{(l)}(u_r^{(l)}, u_z^{(l)})) + \frac{\partial S_{rz}^{(l)}(u_r^{(l)}, u_z^{(l)})}{\partial z} - \frac{S_{\theta\theta}^{(l)}(u_r^{(l)}, u_z^{(l)})}{r} = 0 ; \tag{7}$$

$$\rho \frac{\partial^2 u_z^{(l)}}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}(r S_{zr}^{(l)}(u_r^{(l)}, u_z^{(l)})) + \frac{\partial S_{zz}^{(l)}(u_r^{(l)}, u_z^{(l)})}{\partial z} = 0 ,$$

with the following boundary conditions

$$(u_r^{(l)})_{r=R} = 0 , \ (u_z^{(l)})_{r=R} = 0 ; \tag{8}$$

We have inhomogeneous boundary value problem for determining the complex functions of the wave elastic displacements $u_r^{(n)}, u_z^{(n)}$ of the second nonlinear harmonics normal axisymmetric waves of torsion. It includes an inhomogeneous system of differential equations:

$$\rho \frac{\partial^2 u_r^{(n)}}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}(r S_{rr}^{(l)}(u_r^{(n)}, u_z^{(n)})) + \frac{\partial S_{rz}^{(l)}(u_r^{(n)}, u_z^{(n)})}{\partial z} - \frac{S_{\theta\theta}^{(l)}(u_r^{(n)}, u_z^{(n)})}{r} =$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}(r S_{rr}^{(n)}(u_r^{(l)}, u_z^{(l)})) - \frac{\partial S_{rz}^{(n)}(u_r^{(l)}, u_z^{(l)})}{\partial z} + \frac{S_{\theta\theta}^{(n)}(u_r^{(l)}, u_z^{(l)})}{r} ; \tag{9}$$

$$\rho \frac{\partial^2 u_z^{(n)}}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}(r S_{zr}^{(l)}(u_r^{(n)}, u_z^{(n)})) + \frac{\partial S_{zz}^{(l)}(u_r^{(n)}, u_z^{(n)})}{\partial z} =$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}(r S_{zr}^{(n)}(u_r^{(l)}, u_z^{(l)})) - \frac{\partial S_{zz}^{(n)}(u_r^{(l)}, u_z^{(l)})}{\partial z} ;$$

with boundary conditions of the form

$$(u_r^{(n)})_{r=R} = 0 , \ (u_z^{(n)})_{r=R} = 0 . \tag{10}$$

Solutions of wave equations of linear boundary value problems (7), (8), which describe the axisymmetric longitudinal-shear normal waves in cylinder, can be represented as

$$u_r^{(l)} = -A_1 \tilde{\alpha} J_1(\tilde{\alpha} r) + A_2 ikJ_1(\tilde{\beta} r) ; \tag{11}$$

$$u_z^{(l)} = A_1 ikJ_0(\tilde{\alpha} r) - A_2 \tilde{\beta} J_0(\tilde{\beta} r) ,$$

where $u_r^{(l)}, u_z^{(l)}$ – are complex functions of dynamic displacement in a normal wave with angular frequency $\omega$, wave number $k$ and phase velocity $v$; $v_s = (\mu/\rho)^{1/2}$ – the phase velocity of linear shear waves; $J_n(\gamma r)$ – cylindrical Bessel function; $A$ - an arbitrary amplitude factor. From the boundary conditions of spectral problems (10) and (11) in this case, the dispersion relations was obtained, which define the full spectrum of linear axisymmentrical normal longitudinal-shear waves in the cylinder. These relations have the form

$$k^2 J_0(\sqrt{-k^2+\frac{\Omega^2}{\zeta}}\,r)\,J_1(\sqrt{-k^2+\Omega^2}\,r)+$$

$$\sqrt{-k^2+\Omega^2}\sqrt{-k^2+\frac{\Omega^2}{\zeta}}J_0(\sqrt{-k^2+\Omega^2}\,r)J_1(\sqrt{-k^2+\frac{\Omega^2}{\zeta}}\,r)=0$$

where $\Omega$ is the dimensionless normalized frequency parameter $\Omega=\omega R/v_s$, $\tilde{k}=kR$ – dimensionless normalized wave number; $\tilde{\alpha}^2=\Omega^2/\zeta^2-k^2$; $\tilde{\beta}^2=\Omega^2-k^2$; $\zeta^2=2(1-\nu)/(1-2\nu)$.

The solution of the inhomogeneous boundary value problem (9), (10) on the basis of the algorithm developed analytical transformations was obtained in the form:

$$u_r^{(n)}=-2B_1\tilde{\alpha}J_1(2\tilde{\alpha}r)+B_2ikJ_1(2\tilde{\beta}r)+F_1(r); \tag{12}$$

$$u_z^{(n)}=B_1ikJ_0(2\tilde{\alpha}r)-2B_2\tilde{\beta}J_0(2\tilde{\beta}r)+F_2(r),$$

where $F_1(r)=\sum\limits_{p=1}^{\infty}a_pr^p$; $F_2(r)=\sum\limits_{p=1}^{\infty}b_pr^p$; $B_1=(\chi_{12}F_2(R)-\chi_{22}F_1(R))/(\chi_{11}\chi_{22}-\chi_{21}\chi_{12})$;

$B_2=(\chi_{11}F_2(R)-\chi_{21}F_1(R))/(\chi_{12}\chi_{21}-\chi_{11}\chi_{22})$; $\chi_{11}=-2\tilde{\alpha}J_1(2\tilde{\alpha}r)$; $\chi_{12}=ikJ_1(2\tilde{\beta}r)$; $\chi_{21}=ikJ_0(2\tilde{\alpha}r)$;

$$\chi_{22}=-2\tilde{\beta}J_0(2\tilde{\beta}r).$$

Here $a_p$, $b_p$ - coefficients of the power series, which was obtained as a partial solution of inhomogeneous equations (9). To calculate these coefficients, the following recurrence formulas were received:

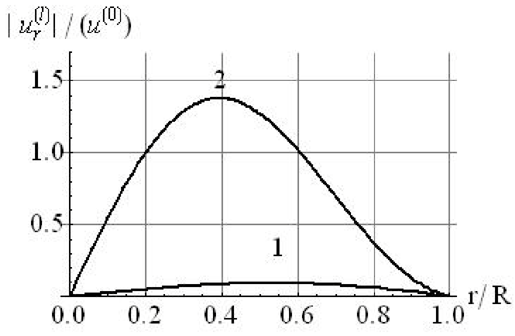
$$a_1=\alpha_1/\Delta_{13}^{(1)};\quad b_1=\beta_1/\Delta_{23}^{(1)};\quad a_2=\frac{\alpha_2-b_1\Delta_{14}^{(1)}}{\Delta_{12}^{(1)}+2\Delta_{13}^{(1)}+2\Delta_{15}^{(1)}};\quad b_2=\frac{\beta_2-a_1(\Delta_{22}^{(1)}-\Delta_{24}^{(1)})}{2\Delta_{23}^{(1)}+2\Delta_{25}^{(1)}};$$

$$a_{p+2}=\frac{\alpha_{p+2}-\Delta_{11}^{(1)}a_p-\Delta_{14}^{(1)}(p+1)b_{p+1}}{\Delta_{12}^{(1)}+\Delta_{13}^{(1)}(p+2)+\Delta_{15}^{(1)}(p+2)(p+1)};\; b_{p+2}=\frac{\beta_{p+2}-\Delta_{21}^{(1)}b_p-\Delta_{22}^{(1)}a_{p+1}-\Delta_{24}^{(1)}(p+1)a_{p+1}}{\Delta_{23}^{(1)}(p+2)+\Delta_{25}^{(1)}(p+2)(p+1)}.$$

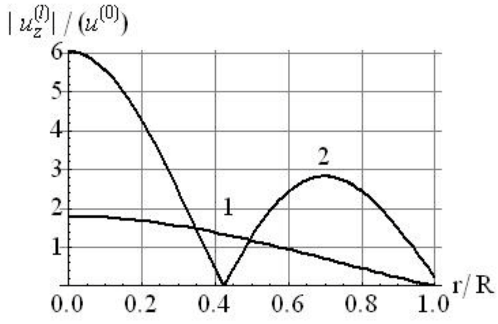
## 2. NUMERICAL RESULTS

Numerical research of nonlinear anharmonic effects was realized for the waveguide of duralumin with following physical and mechanical constants $\rho$=2.79·10$^3$ kg/m$^3$; $\sigma=0.31$; $\mu=2.6\cdot10^{10}\,Pa$; $\lambda=2\sigma\mu/(1-2\sigma)=4.2\cdot10^{10}\,Pa$; $l=-26.46\cdot10^{10}\,Pa$; $m=38.22\cdot10^{10}\,Pa$; $n=36.26\cdot10^{10}\,Pa$.

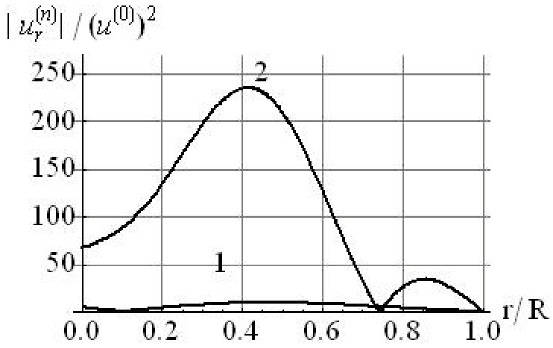
On the base of obtained solutions the radial distribution of dimensionless normalized amplitudes for the wave displacements in the travelling normal longitudinal-shear waves with a relative length and $\lambda = 5R$ from first and second modes of the dispersion spectrum, and for displacements of their non-linear second harmonics were calculated.



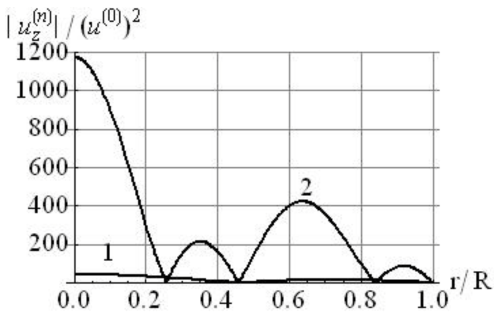

(a) Form of displacement $|u_r^{(l)}|$




(b) Form of displacement $|u_z^{(l)}|$




(c) Form of displacement $|u_r^{(n)}|$

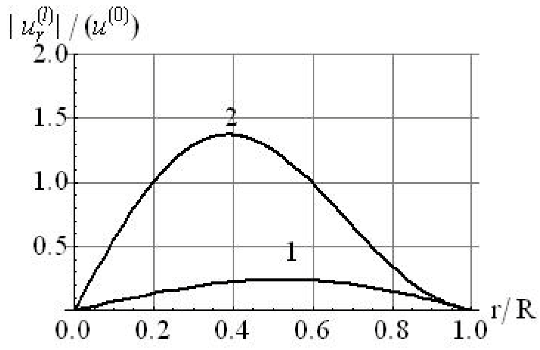



(d) Form of displacement $|u_z^{(n)}|$

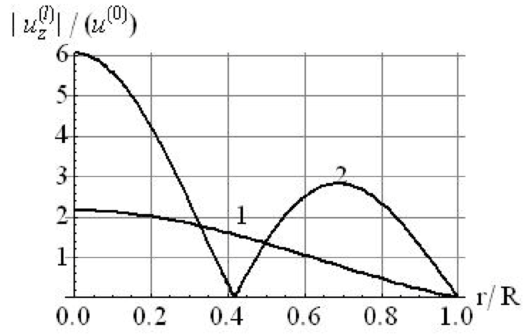
1- the first mode, 2- the second mode

*Fig.1 – Amplitude forms of displacement in first linear and second nonlinear harmonics for the longitudinal-shear waves with $\lambda = 10R$ in a cylinder with fixed boundary.*

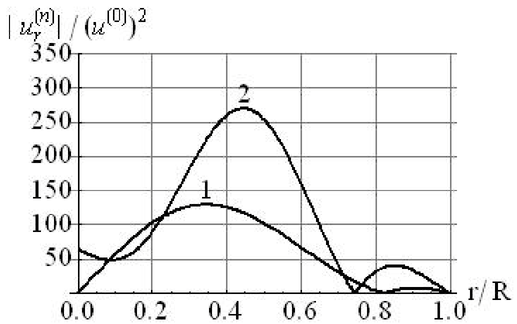
Analysis of the Figures 1-2 allows in particular to make a conclusion about the considerable influence of the parameter of the relative length of the normal waves on the distributions of forms wave displacements along the radial coordinate in the waveguide cross section for nonlinear second harmonic compared to the amplitude of vibrational displacements in the forms of linear normal modes.



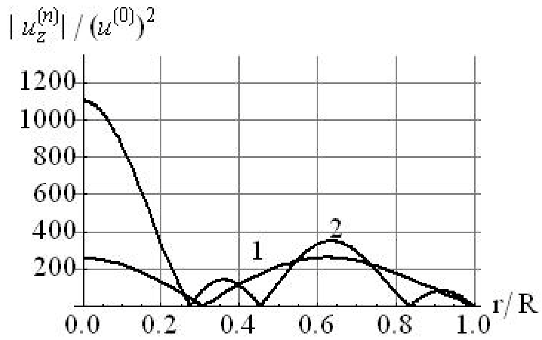

(e) Form of displacement $|u_r^{(l)}|$




(f) Form of displacement $|u_z^{(l)}|$




(g) Form of displacement $|u_r^{(n)}|$




(h) Form of displacement $|u_z^{(n)}|$

1- the first mode, 2- the second mode

*Fig.2 – Amplitude forms of displacement in first linear and second nonlinear harmonics for the longitudinal-shear waves with $\lambda = 5R$ in a cylinder with fixed boundary.*

## CONCLUSIONS

In the article the analytical representations for the second harmonics of the normal longitudinal-shear waves in an isotropic cylinder are obtained for the first time. Some estimates of the amplitudes of nonlinear perturbations for waves of torsion of different relative lengths are presented.

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