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CONSTRUCTING AN OPTIMAL LYAPUNOV FUNCTION FOR INVESTIGATION OF STABILITY OF LINEAR FUZZY HYBRID AUTOMATON

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ABSTRACT

In this paper the condition of α-level stability of linear fuzzy hybrid automaton is converted to a numerical algorithm. A computational procedure that is a hybrid of the Lagrange method and the method of projection of generalized anti-gradient is proposed.

INTRODUCTION

The most common method for investigation of stability of hybrid automata is the method of the Lyapunov functions. General theory of stability of hybrid automata is rather complicated, since the Lyapunov functions needed for investigation of stability should satisfy some complex conditions. For hybrid automata that contain only linear subsystems two approaches are frequently used.

The first of them is based on construction of the Lyapunov quadratic form common for all subsystems. For hybrid automata that have more than two local states there is a theorem: a sufficient condition of existence of the common Lyapunov function is an existence of stable convex combination of matrices A_i, i.e. there are positive α_i, where $\sum_i \alpha_i = 1$, such that matrix $A = \sum_{i=0}^{N} \alpha_i A_i$ is stable [1].

When $N = 2$, this condition is also necessary. But the determination of the convex combination of matrices A_i satisfying this condition is a combinatorial problem with non-linear polynomial complexity. Moreover, there is a large class of systems that don't satisfy this condition, but a stabilizing sequence of switchings exists, and hybrid automaton is stable.

It's shown in [2] that if positive-definite matrices R_i exist, $i = 1..N$ such as $\sum_i (A_i^T R_i + R_i A_i) > 0$, the common quadratic Lyapunov function doesn't exist.

Another approach is a construction of own Lyapunov function for each local state of automaton [3]. This approach assumes a finding of N positive-definite matrices H_i, each of them satisfies its own Lyapunov equation, one symmetric matrix and $2N$ matrices with non-negative elements. These matrices should satisfy a complex system of matrix equations.

In this paper we suggest a constructive approach to check the conditions of stability of linear hybrid automaton. For this we use methods of operational research.

1. OBTAINING AN OPTIMIZATION PROBLEM

We investigate stability of a fuzzy linear hybrid automaton

$$HA = (Q, y, A, B, Init, Inv, Jump), \qquad (1)$$

where

$Q = \{1..N\}$ is a set of local states (discrete variable),

$y \in R^n$ is a continuous variable, changing according to law

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$$y(t) = y(t_i) + \int_{t_i}^{t} Ay(s)ds + \int_{t_i}^{t} By(s)dw(s,x)$$

where $w(s,x)$, $x \in X$ is a process of fuzzy roaming with distribution $\mu(u) = \varphi(\sigma u^2)$ [4], $(X, 2^X, P, N)$ is a *PN* space [5]

$Init = Inv \subseteq \{(q,y) : G_q y \geq 0\}$,

a state of switching (*Jump*) is cyclic ($1 \to 2 \to \ldots \to N \to 1$), continuous ($y(t_i + 0, x) = y(t_i - 0, x)$) and is implemented on hyperplane $y = U_q z$, $z \in R^{n-1}$.

Definition 1. Funnel $y(\bar{y}_0, t, x)$ of fuzzy dynamical system $y(y_0, t, x)$ (not necessarily hybrid automaton) is called α-level stable, if for all $x_0 \in X$ for which $P(\{x_0\}) > \alpha$ for every $\varepsilon > 0$ exists $\delta(\varepsilon)$ such that $|y_0 - \bar{y}_0| < \delta$ implies $|y(y_0, t, x_0) - y(\bar{y}_0, t, x_0)| < \varepsilon$.

Theorem 1 (about the piecewise-quadratic s-function). A linear hybrid automaton (1) is given. If positive-definite matrices H_k (sized $n \times n$) exist such that

$$a_q = \max_{\substack{G_q x \geq 0 \\ x^T x = 1}} x^T (A_q^T H_q + H_q A_q) x + \left| B_q^T H_q + H_q B_q \right| \sqrt{\varphi^{-1}(\alpha)} \sigma < 0$$

and for every switching $q \to (k \bmod N) + 1 = r$ matrix $U_q^T (H_r - H_q) U_q$ is negative-semidefinite, then $x = 0$ is an asymptotically α-level stable stationary point.

So, to check these conditions, we should create an algorithm to obtain matrices H_q that maximally satisfy the theorem. In other words, we should build such matrices H_q that minimize values of a_q. If this minimal value is less than zero, the conditions of the theorem are not fulfilled and we cannot investigate stability of the automaton using the method of the Lyapunov functions. If that value is less than zero, the trivial stationary point is asymptotically stable.

To check stability, we should solve the optimization problem

$$\Phi_1(H) = \max_{\substack{G_q y \geq 0,\, G_q z \geq 0 \\ y^T y = 1,\, z^T z = 1 \\ q \in Q}} \left(y^T (A_q^T H_q + H_q A_q) y + z^T \left(B_q^T H_q + H_q B_q \right) z \sqrt{\varphi^{-1}(\alpha)} \sigma \right) \xrightarrow[H_q]{} \min \qquad (2)$$

with conditions: matrices H_q are positive-definite, for all switchings $q \to r$ matrices $U_q^T (H_r - H_q) U_q$ are seminegative-definite, and elements of matrices $H = \{H_q, q \in Q\}$ are located in some compact domain D that envelops 0. For simplicity of denotes

$$\Psi(q, H, y, z) = y^T (A_q^T H_q + H_q A_q) y + z^T \left(B_q^T H_q + H_q B_q \right) z \sqrt{\varphi^{-1}(\alpha)} \sigma$$

$$L_1 = D \cap \left\{ H : \lambda_{\min}(H_q) \geq 0;\ \lambda_{\max}(U_q^T (H_r - H_q) U_q) \leq 0 \right\}$$

Lemma 1. If function $\psi(H, y) : R^{N_1} \times R^{N_2} \to R$ is continuous and K is a compact on R^{N_2}, then function $\Phi(H) = \min_{y \in K} \psi(H, y)$ is continuous.

Corollary. Minimum and maximum eigenvalues are continuously dependent on coefficients of H.

Theorem 2. Set $L_0 = \left\{ H : \lambda_{\min}(H_q) \geq 0;\ \lambda_{\max}(U_q^T (H_r - H_q) U_q) \leq 0 \right\}$ is a convex closed cone.

Corollary. L_1 is compact.

Theorem 3. Optimization problem (1) has a solution.

Proof. For this we should prove three facts.

1. Function $\Phi_1(H)$ is continuous;
2. There is at least one point $H \in L_1$;
3. Domain L_1 is compact.

Continuity.

$$\Phi_1(H) = \max_{q \in Q} \max_{\substack{G_q y \geq 0,\, G_q z \geq 0 \\ y^T y = 1,\, z^T z = 1}} \left(y^T (A_q^T H_q + H_q A_q) y + z^T \left(B_q^T H_q + H_q B_q \right) z \sqrt{\varphi^{-1}(\alpha)} \sigma \right)$$

Function $\max\limits_{\substack{G_q y\geq 0,\, G_q z\geq 0\\ y^T y=1,\, z^T z=1}}\left(y^T(A_q^T H_q + H_q A_q)y + z^T\left(B_q^T H_q + H_q B_q\right)z\sqrt{\varphi^{-1}(\alpha)\sigma}\right)$ is continuous according to Lemma 1. That's why $\Phi_1(H)$, that is a minimum of finite number of continuous functions, is continuous.

Existence. For sufficiently small γ, point $H = \left\{H_q = \gamma E_n = \{h_{ij}^q = \delta_{ij}\}\right\}$ is located inside L_1. Indeed, for this H limitations $H \in D$, $\lambda_{\max}(H_r - H_q) = 0$ and $\lambda_{\min}(H_q) \geq 0$.

Compactness. Proved above.

These three conditions, according to the Weierstrass theorem, imply existence of solution of the optimization problem.

Theorem 4. Function $\Phi_1(H)$ is convex.

Proof. For this, it's enough to prove convexity of function

$$\Phi_1^q(H) = \max_{\substack{G_q y\geq 0,\, G_q z\geq 0\\ y^T y=1,\, z^T z=1}}\left(y^T(A_q^T H_q + H_q A_q)y + z^T\left(B_q^T H_q + H_q B_q\right)z\sqrt{\varphi^{-1}(\alpha)\sigma}\right)$$

So, given H' and H'', denote $H = \gamma H' + (1-\gamma)H''$.

$\Phi_1^q(H)$ is essentially $\max\limits_{y,z} L(H, y, z)$, where L is a linear functional of *H*. Then $\Phi_1^q(\alpha H_1 + \beta H_2) \leq \Phi_1^q(\alpha H_1) + \Phi_1^q(\beta H_2) = \alpha\Phi_1^q(H_1) + \beta\Phi_1^q(H_2)$, when $\alpha, \beta \geq 0$. So, $\Phi_1^q(H)$ is convex.

2. METHOD OF NUMERICAL SOLUTION

As it was said above, there are three limitations for the coefficients h_{ij}^q: $H \in D$, $\lambda_{\min}(H_q) \geq 0$, $\lambda_{\max}(U_q^T(H_r - H_q)U_q) \leq 0$. For implementation of the first condition we can use projection of gradient, if we pick specially-shaped D. For the second – the gradient is projected as $H_q' = H_q + \lambda_{\min}(E_n)$. And for the third one it's impossible to project the gradient. So, we use a hybrid of Lagrange method and gradient projection method. We construct next Lagrange function:

$$\Phi(H) = \Phi_1(H) + \sum_{\substack{q\to r\\ q,r\in Q}} \theta^{qr}\Phi^{qr}(H)$$

where $\Phi^{qr}(H) = \lambda_{\max}(U_q^T(H_r - H_q)U_q)$. Let us assume $D = \left\{\left|h_{ij}^q\right| \leq 1\right\}$.

Definition 2. Generalized gradient of function $\Phi(x)$ is a vector $\nabla^*(x)$ such that $\Phi(z) - \Phi(x) \geq (\nabla^*(x), z - x)$.

Theorem 5. The following equation is a generalized gradient of $\Phi_1(H)$:

$$\nabla_{\Phi_1}^*(H) = \left\{h_{ij}^q = y_0^T(A^T\Delta_{ij} + \Delta_{ij}A)y_0 + z_0^T(A^T\Delta_{ij} + \Delta_{ij}A)z_0\right\}$$

where y_0 and z_0 are n-dimensional vectors that realize maximum of function Ψ; Δ_{ij} is a matrix $n \times n$ that has one unit element on intersection of *i* and *j*.

The only remaining thing is finding the generalized gradient of the function $\Phi^{kl}(H) = \lambda_{\max}(U_k^T(H_\ell - H_k)U_k)$.

Theorem 6. Equation $\nabla_{\Phi^{k\ell}}^*(H) = \{h_{ij}^q\}$, where $h_{ij}^q = \begin{cases} 0;\ q \neq k, \ell \\ -u_0^T U_k^T \Delta_{ij} U_k u_0;\ q = k \\ u_0^T U_k^T \Delta_{ij} U_k u_0;\ q = \ell \end{cases}$; u_0 is a vector of dimension $n-1$ and norm 1 that realizes maximum of $\left\| u^T U_k^T(H_\ell - H_k)U_k u\right\|$, is a generalized gradient of function $\Phi^{k\ell}(H)$.

Proof.

$$\Phi^{k\ell}(H_1) - \Phi^{k\ell}(H_0) = \lambda_{\max}(U_k^T(H_1^\ell - H_1^k)U_k) - \lambda_{\max}(U_k^T(H_0^\ell - H_0^k)U_k)$$

The matrix $U_k^T(H_1^\ell - H_1^k)U_k$ is positive-definite, so one has

$$\lambda_{\max}(U_k^T(H_1^\ell - H_1^k)U_k) = \max_{\|u\|=1} u^T U_k^T(H_\ell - H_k)U_k u$$

Thus, the following holds

$$\Phi^{k\ell}(H_1)-\Phi^{k\ell}(H_0)=u_1^T U_k^T (H_1^\ell - H_1^k)U_k u_1 - u_0^T U_k^T (H_0^\ell - H_0^k)U_k u_0$$

where $u_0 = \arg\max_{\|u\|=1} u^T U_k^T (H_0^\ell - H_0^k)U_k u$, u_1 is the same for matrix H_1.

This equation may be rewritten as

$$\Phi^{k\ell}(H_1)-\Phi^{k\ell}(H_0)=u_1^T U_k^T (H_1^\ell - H_1^k)U_k u_1 - u_0^T U_k^T (H_1^\ell - H_1^k)U_k u_0$$
$$+u_0^T U_k^T \left[(H_1^\ell - H_1^k)-(H_0^\ell - H_0^k)\right]U_k u_0.$$

Because maximum for matrix H_1 holds for vector u_1,

$$u_1^T U_k^T (H_1^\ell - H_1^k)U_k u_1 - u_0^T U_k^T (H_1^\ell - H_1^k)U_k u_0 \ge 0$$

Then

$$\Phi^{k\ell}(H_1)-\Phi^{k\ell}(H_0)\ge u_0^T U_k^T \left[(H_1^\ell - H_1^k)-(H_0^\ell - H_0^k)\right]U_k u_0$$

We get a definition of generalized gradient:

$$\Phi^{k\ell}(H_1)-\Phi^{k\ell}(H_0)\ge (\nabla^*_{\Phi^{k\ell}}(H_0), H_1 - H_0)$$

2.1 Movement down the "canyon"

Sometimes a situation happens when on some step a maximum of function Ψ holds simultaneously for two different q: $q=q_0$ and $q=q_1$. We can call this situation a "canyon". Target function in the "canyon" is continuous, but it has a discontinuous derivative, so in general case the generalized gradient may not exist. For simplicity let us rewrite our problem as

$$\Phi(H)=\max_{y,q}\Psi(H,y,q)\to \min$$

One denotes: $\Phi(H,q)=\max_{y}\Psi(H,y,q)$.

Calculating a generalized gradient of the function $\Phi(H)$, in reality we calculate a generalized gradient of all $\Phi(H,q)$. By definition of a generalized gradient, it defines a semi-space Ω, for which for every $H_1\in\Omega$ (close enough to H) holds $\Phi(H_1,q)<\Phi(H,q)$. If we intersect the subspaces that correspond to $q=q_0$ and $q=q_1$, we obtain an infinite pyramid that corresponds to all possible movements from current point H.

Number of "blocking" q is always less than number of free variables in H. That's why the set of possible direction is non-empty. We can find at least one element of intersection of mentioned semi-spaces from the system $\left\langle \nabla_i^*, H_1 \right\rangle \le 0, i\in\{0,1\}$.

The target function is uniform ($\Phi(kH)=k\Phi(H)$). That's why we treat the solution as optimal, when on the next step we are on the boundary of D, and because of "canyon" limitation we cannot move without moving beyond $\left|h_{ij}^q\right|\le 1$.

2.2 Computational procedure

First treat all variables of matrix H as "unlocked". Repeat the procedure:

1. Compute $\nabla^*_\Phi(H,\theta)$. If the maximum holds for several $q\in Q$, compute generalized gradient for all such q and find a vector that is in the intersection of subspaces.

2. For all "locked" variables: if the corresponding coordinate of generalized anti-gradient ∇ leads inside cube $\left|h_{ij}^q\right|\le 1$, "unlock" the variable. If not, replace the coordinate of anti-gradient with zero. If after these limitations the anti-gradient turns to zero, STOP: no solution found.

3. Find ρ, for which $H-\rho\nabla$ doesn't move beyond $\left|h_{ij}^q\right|\le 1$. If $\Phi(H-\rho\nabla)<\Phi(H)$, set $H:=H-\rho\nabla$ and "lock" the coordinate that became a limitation.

4. If $\Phi(H-\rho\nabla)\ge\Phi(H)$, find optimal ρ according to rules of gradient method. Assign: $H:=H-\rho\nabla$.