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DYNAMICS OF A TIMOSHENKO BEAM ON AN ELASTIC FOUNDATION

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ABSTRACT

Free transverse waves in Timoshenko beam resting on Winkler foundation are studied. Dispersion curves are derived and analyzed in generalized dimensionless variables and parameters. Special attention is paid to clearing up the physical sense of the second spectrum of Timoshenko beam.

INTRODUCTION

There are two main reasons for studying dynamics of Timoshenko beam (TB, Timoshenko [1, 2]). From *theoretical point of view*, the Timoshenko model has certain advantages over the classical Euler-Bernoulli (E-B) model. It is known that E-B model has non-wave character (according to E-B dynamic equation, a perturbation propagates along the beam with infinite large velocity, see, e.g., Uflyand [3]. The Timoshenko model results in an equation of wave character.

From *practical viewpoint*, Timoshenko beam model, especially in case of elastic foundation, is of great interest in view of the development of the high-velocity transport. The action of moving loads often gives rise to localized stress-strain states for which shear deformability should be taken into account.

Our interest to dynamics of Timoshenko beam was caused by the following particular problem. It is known that in the TB for each wave number there exist two natural frequencies, and so two spectra of oscillations can be separated. During last decades discussion continued about the meaning of the second spectrum (see, e. g., [4-9] and for review - Stephen [9]), and many investigators adhere to opinion that “the second spectrum predictions of TB theory should be disregarded” [9].

In this paper an analysis of free transverse waves in TB on Winkler foundation is carried out. The use of dimensionless variables and parameters (Manevich A. [10]) make it possible to draw general relations and conclusions. One of the main goals of this paper is to show that when we consider the TB *on elastic foundation* we obtain new convincing proofs of necessity and validity of the second branch of the spectrum.

1. GOVERNING EQUATIONS

Equations of motion for TB on the Winkler foundation are derived using known hypotheses. Deformations of the beam are described by two independent functions – the angle of the cross section rotation ψ and the shear angle γ (at the neutral axis). The total slope of the bent axis is

$$\frac{\partial y}{\partial x} = \psi + \gamma \tag{1}$$

where $y(x,t)$ is the transverse displacement. The longitudinal displacement of a point on distance z from the neutral axis and the longitudinal deformations are expressed via angle ψ: $u = -z\psi$, $\varepsilon_x = -z\,\partial\psi/\partial x$.

The bending moment and the transverse shear force in the cross section are specified by known expressions:

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$$M = -EJ\frac{\partial \psi}{\partial x}, \qquad Q = k'A\tau = k'AG\left(\frac{\partial y}{\partial x} - \psi\right) \tag{2}$$

where k' is the coefficient which depends upon the cross section shape (see, e.g., [2]), A and J are the cross section area and the moment of inertia, E and G are moduli of elasticity in tension and shear, respectively.

Equations of the force balance for a beam loaded by a transverse load $q_0(x,t)$ and resting on the elastic foundation with stiffness factor w_0 are:

$$\frac{\partial Q}{\partial x} - \rho A\frac{\partial^2 y}{\partial t^2} + q_0(x,t) - w_0\, y = 0\,, \qquad -\rho J\frac{\partial^2 \psi}{\partial t^2} + Q - \frac{\partial M}{\partial x} = 0 \tag{3}$$

These equations with account of the above relations result in two differential equations of motion in y and ψ:

$$k'GA\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x} - \psi\right) - \rho A\frac{\partial^2 y}{\partial t^2} + q_0(x,t) - w_0\, y = 0 \tag{4}$$

$$EJ\frac{\partial^3 \psi}{\partial x^3} - \rho J\frac{\partial^3 \psi}{\partial x\,\partial t^2} + \rho A\frac{\partial^2 y}{\partial t^2} - q_0(x,t) + w_0\, y = 0 \tag{5}$$

Excluding the angle ψ we obtain the single equation with respect to the displacement $y(x,t)$:

$$\begin{aligned} &EJ\frac{\partial^4 y}{\partial x^4} - \rho J\left(1 + \frac{E}{k'G}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} + \frac{\rho^2 J}{k'G}\frac{\partial^4 y}{\partial t^4} + \rho A\frac{\partial^2 y}{\partial t^2} + w_0\left[1 + \frac{\rho J}{k'GA}\frac{\partial^2}{\partial t^2} - \frac{EJ}{k'GA}\frac{\partial^2}{\partial x^2}\right]y = \\ &= \left[1 + \frac{\rho J}{k'GA}\frac{\partial^2}{\partial t^2} - \frac{EJ}{k'GA}\frac{\partial^2}{\partial x^2}\right]q_0 \end{aligned} \tag{6}$$

For the free oscillation problem the right hand side in (6) is equal to zero. The boundary conditions for the set (4), (5) in variables y and ψ can be derived utilizing the Hamilton's principle.

Let us introduce dimensionless variables and parameters (Manevich A. [10]):

$$\xi = \frac{x}{r_0},\ \ Y = \frac{y}{r_0}\ ,\ \ \tau = \frac{c}{r_0}t\ ,\ c^2 = \frac{E}{\rho},\ \ r_0^2 = \frac{J}{A}\ \ \chi = \frac{E}{k'G},\ \ q = \frac{q_0 r_0}{E\,A},\ \ w = \frac{w_0 r_0^2}{E\,A} \tag{7}$$

Here c is the sound velocity in the beam material, r_0 is the cross section radius of gyration, χ is the shear deformability parameter, w is the foundation stiffness parameter. Note that for classical Euler-Bernoulli and Rayleigh models $\chi = 0$, that corresponds to infinitely large shear stiffness.

In variables (7) equations (4), (5) take the form

$$\frac{\partial}{\partial \xi}\left(\frac{\partial Y}{\partial \xi} - \psi\right) - \chi\frac{\partial^2 Y}{\partial \tau^2} + \chi\, q\ (\xi,\tau) - \chi\, wY = 0 \tag{8}$$

$$\frac{\partial^3 \psi}{\partial \xi^3} - \frac{\partial^3 \psi}{\partial \xi\,\partial \tau^2} + \frac{\partial^2 Y}{\partial \tau^2} - q\ (\xi,\tau) + wY = 0 \tag{9}$$

and equation (6) transforms into

$$\cdot\frac{\partial^4 Y}{\partial \xi^4} - (1+\chi)\frac{\partial^4 Y}{\partial \xi^2\,\partial \tau^2} + \chi\frac{\partial^4 Y}{\partial \tau^4} + \frac{\partial^2 Y}{\partial \tau^2} = \left(1 + \chi\frac{\partial^2}{\partial \tau^2} - \chi\frac{\partial^2}{\partial \xi^2}\right)(q - wY) \tag{10}$$

This equation includes only two generalized parameters χ и w, characterizing the shear deformability and the foundation stiffness. From (8)-(10) one can obtain equations for particular cases of free waves and oscillations ($q=0$); for beam without elastic foundation ($w=0$, this case on the base of the dimensionless equations was considered in [10]). If $\chi=0$ one obtains dimensionless equations for the Rayleigh model, which differs from the classical Euler-Bernoulli model (E-B) with accounting the rotatory inertia of beam (for the E-B beam on the Winkler foundation the second term in left hand side of (10) vanishes).

The obtained equations are apparently preferable in comparison with often used dimensionless equations with several parameters.

The angle ψ can be expressed via Y using equation (9). For the derivative ψ_ξ one has relationship

$$\frac{\partial^2 Y}{\partial \tau^2}+wY=\frac{\partial^2 \psi_\xi}{\partial \tau^2}-\frac{\partial^2 \psi_\xi}{\partial \xi^2}+q\ (\xi,\tau) \tag{11}$$

The shear angle $\gamma=\dfrac{\partial y}{\partial x}-\psi=\dfrac{\partial Y}{\partial \xi}-\psi$ and its derivative γ_ξ is expressed via dimensionless variables Y and ψ_ξ:

$$\gamma_\xi \equiv \frac{\partial \gamma}{\partial \xi}=\frac{\partial^2 Y}{\partial \xi^2}-\psi_\xi \tag{12}$$

2. SOLUTION FOR FREE WAVES IN INFINITELY LONG BEAM

Here we consider only free waves in infinitely long beam ($q=0$), which are described by equation

$$\frac{\partial^4 Y}{\partial \xi^4}-(1+\chi)\frac{\partial^4 Y}{\partial \xi^2\,\partial \tau^2}+\chi\frac{\partial^4 Y}{\partial \tau^4}+\frac{\partial^2 Y}{\partial \tau^2}+w\left(1+\chi\frac{\partial^2}{\partial \tau^2}-\chi\frac{\partial^2}{\partial \xi^2}\right)Y=0 \tag{13}$$

A solution is seeking in the form of harmonic waves

$$Y(\xi,\tau)=Y_0\,e^{i(k\xi-\omega\tau)} \tag{14}$$

Substitution of (14) into (13) gives the frequency equation

$$\chi\omega^4-\omega^2\left[1+(\chi+1)k^2+w\chi\right]+w\left(1+\chi k^2\right)+k^4=0 \tag{15}$$

The roots are:

$$\omega_{1,2}^2=\frac{1}{2\chi}\left[1+(\chi+1)k^2+w\chi\mp\sqrt{D}\right] \tag{16}$$

where

$$D=\left(1+(\chi+1)k^2+w\chi\right)^2-4\chi\left(w(1+\chi k^2)+k^4\right) \tag{17}$$

It can be readily seen that the both roots of equation (16) are real and positive. These two eigenvalues ω_1 and ω_2 for a given wave number k determine two phase velocities for each k: $v_{\phi,1,2}=\omega_{1,2}/k$. The existence of two branches, or two spectra, is a principal distinction of the Timoshenko model from the E-B and Rayleigh models, which was revealed for beam without foundation in early papers ([3] and others).

Let us find relations between amplitudes of the transverse deflections Y and amplitudes of angles ψ_ξ and γ_ξ for each the branch. Assuming these quantities in view of (11), (12) in the form

$$Y_j(\xi,\tau)=Y_{0j}\,e^{i(k\xi-\omega_j\tau)},\ \psi_{\xi,j}(\xi,\tau)=\psi_{\xi,0j}\,e^{i(k\xi-\omega_j\tau)},\ \ \gamma_{\xi,j}(\xi,\tau)=\gamma_{\xi,0j}\,e^{i(k\xi-\omega_j\tau)}\ (j=1,2)\quad(18)$$

and substituting into (11) with $q=0$ and (12), one obtains

$$\psi_{\xi,0j}=\frac{\omega_j^2-w}{\omega_j^2-k^2}Y_0,\quad \gamma_{\xi,0j}=\frac{k^4-\omega_j^2k^2-\omega_j^2+w}{\omega_j^2-k^2}Y_0\quad (j=1,2)\qquad(19)$$

The following identity follows from the frequency equation (15):

$$k^4-\omega_j^2k^2-\omega_j^2+w=\chi\left(\omega_j^2-k^2\right)\left(w-\omega_j^2\right)\quad (j=1,2)$$

Then the second relation (19) (with account of the first one) yields

$$\gamma_{\xi,0j}=-\chi\left(\omega_j^2-w\right)Y_{0j}=-\chi\,(\omega_j^2-k^2)\,\psi_{\xi,0j}\quad (j=1,2)\qquad(20)$$

3. ANALYSIS OF THE SOLUTION

Consider first the simplest *limit case of long waves* $k\to 0$. Putting in (16) $k=0$, one has

$$\omega_{1,2}^2(k=0)=\frac{1}{2\chi}\left[1+w\chi\mp\left|1-w\chi\right|\right]\qquad(21)$$

This yields

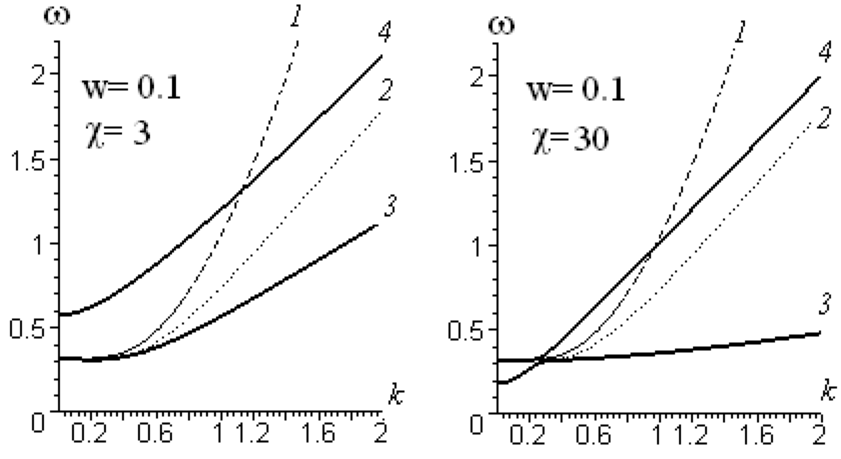
$$\text{for } w\chi<1:\quad \omega_1^2(k=0)=w,\ \ \omega_2^2(k=0)=\frac{1}{\chi}\qquad(22,a)$$

$$\text{for } w\chi>1:\quad \omega_1^2(k=0)=\frac{1}{\chi},\ \ \omega_2^2(k=0)=w\qquad(22,б)$$

At changing stiffness of the elastic foundation to the beam shear stiffness ratio the first and the second spectrum “change” with their limit points (or with analytical dependencies of these stiffness’s). The first branch in limit $k\to 0$ is determined by the smaller of these stiffness’s, the second one – by the larger of the stiffness’s. The case $w\chi<1$ can be named “weak foundation”, and $w\chi>1$ – «strong foundation».

If k is small (but not equal to 0), then due to continuous analytical dependence (16) value of ω_1^2 is close to w for weak foundation ($w<1/\chi$), and to $1/\chi$ for strong one ($w>1/\chi$). For ω_2^2 the picture will be opposite.

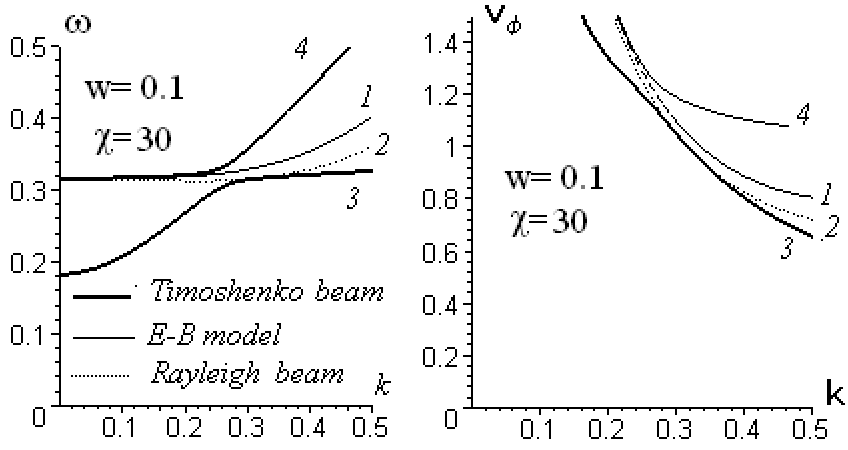
In Fig. 1 dispersion curves for frequency are presented for $w=0.1$ in two cases: $\chi=3$ (*a*) and $\chi=30$ (*б*). Two branches for TB are constructed (bold curves), and for comparison curves for E-B model (curves 1) and Rayleigh model (curves 2) are given. Portions of the curves for relatively small k are shown on a large scale in Fig, 2, *a*.



a *b*

Fig. 1 Dispersion curves for frequency $\omega - k$ for beam on elastic foundation $w = 0{,}1$ in cases $\chi = 3$ (a) and $\chi = 30$ (б). Two branches for TB (bold curves), curves for E-B (1) and Rayleigh models (2)

In Fig. 1,a (the case of "weak" foundation) the first branch for TB (curve 3) has the same asymptotics for small k, as do the E-B and Rayleigh models (curves 1, 2), and these curves practically merge for $k < 0.15$. But the second branch (curve 4) at large k is close to the Rayleigh model. Note that the dispersion curves differ from those for case of beam without foundation ($w = 0$) only with "shifting" their left parts (curves 1, 2, 3 originate from one of points (22,a) or (22, *b*), not from zero point).



а *б*

Fig. 2. Dispersion curves for frequency $\omega - k$ (a) and phase velocity $v_\phi - k$ (б) for beam on elastic foundation, $w = 0.1$, $\chi = 30$, on a large scale. Two branches for TB (curves 3, 4), curves for E-B (1) and Rayleigh models (2)

But for "strong" foundation we see another picture (Fig. 1,*b*, Fig, 2, *a*). Both the branches of TB (curves 3, 4) consist of two portions. The left portion of the first branch is a continuation of the right part of the second branch, and inversely. In other words, both branches "have changed" with their portions. This peculiarity is underlined by the fact that curves for E-B and Rayleigh models (1, 2) now approach the second branch of TB model at $k \to 0$ (not the first branch, as they do for "weak" foundation).

In Fig. 2, *b,* dispersion curves for phase velocity $v_\phi - k$ are shown for moderately small k values, which demonstrate the same behavior as Fig. 2, *a*. We also see a twisting point on curve 3 (first branch for TB), and curves 1, 2, which were close to the curve 3, begin for small k to approach curve 4 – the second TB branch.

For elucidating the physical meaning of the second spectrum of TB let us now note that *for beam without foundation* one has $\omega_1^2 - k^2 < 0$ (*j*=1, upper sign) and $\omega_2^2 - k^2 > 0$, *j*=2, lower sign (it can be proved using (16), (17)). Then it follows from (20) that for the first branch oscillations of γ_ξ (dimensionless curvature of the beam due to shear) occur in phase with oscillations of ψ_ξ (dimensionless curvature due to bending), and that for the second branch these oscillations occur in anti-phase.

The similar statements are also valid for the angles of shear γ and bending ψ. Thus, *the first spectrum of natural frequencies for beam without foundation relates to wave (oscillation) modes for which the angle of rotation of the cross section and of shear angle oscillate in phase; the second spectrum relates to waves for which these angles oscillate in anti-phase.*

It can be also proved that the similar statement is valid for the *beam on elastic foundation* in the case of "weak foundation".

Eigenvector $(\psi_{\xi 0}, \gamma_{\xi 0})$ (and (ψ_0, γ_0)) in limit $k \to 0$ is determined by (22). For the first branch this vector is (ψ_0, γ_0)= (1, $-\chi w$) in the case $w < 1/\chi$, and $(\psi_{\xi 0}, \gamma_{\xi 0}) = (1, -1)$ in the case $w > 1/\chi$. For the second branch limit eigenvector is (ψ_0, γ_0)= $(1, -1)$ at $w < 1/\chi$ and (ψ_0, γ_0)=(1, $-\chi w$) at $w > 1/\chi$. We see again that at transition of w value through $1/\chi$ the eigenvectors "change" occurs. This demonstrates the equivalence of two branches for the TB and refutes statement that the second spectrum of TB is "unphysical".

CONCLUSIONS

The presented analysis of free transverse waves in Timoshenko beam on the Winkler foundation based on dimensionless equations with two generalized parameters allows to draw principal conclusions concerning dynamics of TB. The obtained solution, in particular, brings to light the meaning of the second spectrum of TB. It is shown that both the spectra are equivalent in certain sense that refutes the view on the second branch as "unphysical" one.

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