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INVESTIGATION OF DYNAMICAL INSTABILITY OF VIBRATION MODES FOR LAMINATED PLATES AND SHALLOW SHELLS

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ABSTRACT

Research method of the dynamic stability of the geometrically nonlinear vibration modes of the shallow shells with complex plan form is used. Mathematical statement of the problem is carried out in framework of reinforced theory shells of the first order. The proposed method is based on the R-functions theory, variational methods, "limited stability criterion", obtained from the definition stability by Lyapunov and method by Runge-Kutta. Numerical results for shells with complex form under transverse periodic load can be received using the realized software of designed numerically-analytical approach.

INTRODUCTION

Laminated composite shallow shells are widely applied as structure components in the construction of aerospace, mechanical, ship-building and other branches. Dynamical instability analysis of composite shallow shells and plates subjected to a harmonic transverse load has received considerable attention in the literature [1, 2, 3, 4]. However the most papers consider an investigation of nonlinear vibrations of plates and shells with simple enough form. There are only few works in which laminated shallow shells or plates with a shape different from rectangular, circle or ellipse are presented. Deficiency of such papers is explained by the difficulties of construction of analytical expressions for basic functions. These functions are needed to reduce a nonlinear system of differential equations with partial derivatives to a system of the ordinary differential equations for time. One of universal approaches, which can be used for solving this problem, is founded on the application of the R-functions theory [5, 6]. This theory allows a construction of complete set of the coordinate functions for different types of boundary conditions. In this paper the R-function theory together with variational methods and the "limited criterion of stability by Lyapunov" [7] is applied as a new approach to investigate vibration modes of laminated shallow shells supported on complex plan form.

1. THE GOVERNING EQUATION OF THE SHALLOW SHELLS THEORY

Consider a laminated shallow shell of an arbitrary plan form constructed of a finite number N of orthotropic layers, oriented arbitrary with respect to the shell coordinates (x,y,z). In this paper we shall only investigate symmetric laminated shallow shells. The components of the displacements at an arbitrary point of the shell in the x,y and z directions are u,v and w respectively. According to the first-order shear deformation theory, the inplane displacements u and v are linear functions of the coordinate z and the transverse displacement w is a constant throughout the thickness of the shell. Under this assumption the displacement field may be given in the following form:

$$u' = u + z\psi_x,\ v' = v + z\psi_y,\ w' = w$$

where u, v and w are the displacements at the middle surface, ψ_x and ψ_y are the rotations of the middle surface about the Oy and Ox axes respectively. The nonlinear strain-displacement relations of the shallow shells can be written as

$$\varepsilon'_{11} = \varepsilon_{11} + zk_x,\quad \varepsilon'_{22} = \varepsilon_{22} + zk_y,\quad \varepsilon'_{33} = 0,\quad \varepsilon'_{12} = \varepsilon_{12} + zk_{xy},\quad \varepsilon'_{13} = w_{,x} - uk_1 + \psi_x$$

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$$\varepsilon'_{23} = w_{,y} - vk_2 + \psi_y\,, \quad \varepsilon'_{xz} = \psi_x + w_{,x}\,, \quad \varepsilon'_{yz} = \psi_y + w_{,y}$$

in which

$$\varepsilon_{11} = u,_x + k_1 w + \frac{1}{2} w,_x^2, \quad \varepsilon_{22} = v,_y + k_2 w + \frac{1}{2} w,_y^2, \quad \varepsilon_{12} = u,_x + v,_y + w,_x\, w,_y$$

$$k_x = \psi_{x,x}\,, \quad k_y = \psi_{y,y}\,, \quad k_{xy} = \psi_{x,y} + \psi_{y,x}$$

Here k_1 and k_2 are two principal curvatures of shallow shells, the subscripts following a comma stand for partial differentiation. Let us present unknown functions as components of the following vector $U = (u, v, w, \psi_x, \psi_y)^T$, then the governing equations are derived as follows:

$$LU = -Nl(w) + m\frac{\partial^2 U}{\partial t^2} + P \tag{1}$$

where

$$L = \lfloor L_{ij} \rfloor_{i=\overline{1,5},\, j=\overline{1,5}}\,, \quad Nl(w) = (Nl_1(w), Nl_2(w), Nl_3(w), 0, 0)^T$$

$$m = (m_1, m_1, m_1, m_2, m_2)^T\,, \quad P = (0, 0, q(x, y, t), 0, 0)^T$$

Here L_{ij} are linear differential operators, which can be expressed as follows:

$$L_{11} = C_{66}\frac{\partial^2}{\partial x^2} + 2C_{16}\frac{\partial^2}{\partial x \partial y} + C_{66}\frac{\partial^2}{\partial y^2} - k_1^2 C_{55}\,\cdot,$$

$$L_{22} = C_{66}\frac{\partial^2}{\partial x^2} + 2C_{26}\frac{\partial^2}{\partial x \partial y} + C_{22}\frac{\partial^2}{\partial y^2} - k_2^2 C_{44}$$

$$L_{12} = L_{21} = C_{16}\frac{\partial^2}{\partial x^2} + (C_{12} + C_{66})\frac{\partial^2}{\partial x \partial y} + C_{26}\frac{\partial^2}{\partial y^2} - k_1 k_2 C_{45}$$

$$L_{13} = -L_{31} = \left((k_1 C_{11} + k_2 C_{12} + k_1 C_{55})\frac{\partial}{\partial x} + (k_1 C_{16} + k_2 C_{26} + k_1 C_{45})\frac{\partial}{\partial y}\right)$$

$$L_{14} = L_{41} = k_1 C_{55}\,, \quad L_{15} = L_{51} = k_2 C_{44}$$

$$L_{23} = -L_{32} = \left((k_1 C_{16} + k_2 C_{26} + k_2 C_{45})\frac{\partial}{\partial x} + (k_1 C_{12} + k_2 C_{22} + k_2 C_{44})\frac{\partial}{\partial y}\right)$$

$$L_{24} = L_{42} = k_2 C_{45}\,, \quad L_{25} = L_{52} = k_2 C_{44}$$

$$L_{33} = C_{55}\frac{\partial^2}{\partial x^2} + 2C_{45}\frac{\partial^2}{\partial x \partial y} + C_{44}\frac{\partial^2}{\partial y^2} - (C_{11}k_1^2 + 2C_{12}k_1 k_2 + C_{22}k_2^2)$$

$$L_{34} = L_{43} = C_{55}\frac{\partial}{\partial x} + C_{45}\frac{\partial}{\partial y}\,, \quad L_{35} = -L_{53} = C_{45}\frac{\partial}{\partial x} + C_{44}\frac{\partial}{\partial y}$$

$$L_{44} = D_{11}\frac{\partial^2}{\partial x^2} + 2D_{16}\frac{\partial^2}{\partial x \partial y} + D_{66}\frac{\partial^2}{\partial y^2} - C_{55}$$

$$L_{45} = L_{54} = \left(D_{16}\frac{\partial^2}{\partial x^2} + (D_{12} + D_{66})\frac{\partial^2}{\partial x \partial y} + D_{26}\frac{\partial^2}{\partial y^2} - C_{44}\right)$$

$$L_{55} = D_{66}\frac{\partial^2}{\partial x^2} + 2D_{26}\frac{\partial^2}{\partial x \partial y} + D_{22}\frac{\partial^2}{\partial y^2} - C_{44}$$

The right part of the equation (1), that is $Nl(w)$ is presented as follows:

$$Nl_1(w) = \frac{\partial}{\partial x}\left(\frac{1}{2}C_{11}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}C_{12}\left(\frac{\partial w}{\partial y}\right)^2 + C_{16}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) +$$

$$+ \frac{\partial}{\partial y}\left(\frac{1}{2}C_{16}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}C_{26}\left(\frac{\partial w}{\partial y}\right)^2 + C_{66}\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right),$$

$$Nl_2(w)=\frac{\partial}{\partial x}\left(\frac{1}{2}C_{16}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}C_{26}\left(\frac{\partial w}{\partial y}\right)^2+C_{66}\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right)+$$

$$+\frac{\partial}{\partial y}\left(\frac{1}{2}C_{12}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}C_{22}\left(\frac{\partial w}{\partial y}\right)^2+C_{26}\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right),$$

$$Nl_3(u,v,w,\psi_x,\psi_y)=\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2(k_1C_{11}+k_2C_{12})+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2(k_1C_{12}+k_2C_{22})+$$

$$+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}(k_1C_{16}+k_2C_{26})+N_{11}\frac{\partial^2 w}{\partial x^2}+N_{22}\frac{\partial^2 w}{\partial y^2}+2N_{12}\frac{\partial^2 w}{\partial x\partial y}$$

The coefficients C_{ij}, D_{ij} $(K_{ij}=0)$ and m_j, $(j=\overline{1,5})$ are calculated by following formulas:

$$(C_{ij},D_{ij})=\sum_{m=1}^{n}\int_{h_m}^{h_{m+1}}B_{ij}^{(m)}(1,z^2)dz,\ (i,j=1,2,6),\quad C_{ij}=k_i^2\sum_{m=1}^{n}\int_{h_m}^{h_{m+1}}B_{ij}^{(m)}dz,\quad (i,j=4,5)$$

$$m_j=\sum_{m=1}^{n}\int_{h_m}^{h_{m+1}}\rho_m dz,\ (j=1,2,3),\quad m_j=\sum_{m=1}^{n}\int_{h_m}^{h_{m+1}}\rho_m z^2dz,\ (j=4,5)$$

Here $B_{ij}^{(m)}$ are the stiffness coefficient of the m-th layer, k_i $(i=4,5)$ is the shear correction factors.

2. SOLUTION METHOD

Let us present unknown functions with help series using the eigenfunctions $\vec{U}_i=(u_i,v_i,w_i,\psi_{x_i},\psi_{y_i})$ of the corresponding linear vibration problem

$$w=\sum_{i=1}^{2}y_i(t)w_i(x,y),\quad \psi_x=\sum_{i=1}^{2}y_i(t)\psi_{x_i}(x,y),\quad \psi_y=\sum_{i=1}^{2}y_i(t)\psi_{y_i}(x,y)$$

$$u=\sum_{i=1}^{2}y_i(t)u_i(x,y)+\sum_{i=1}^{2}\sum_{j=}^{2}y_iy_ju_{ij},\quad v=\sum_{i=1}^{2}y_i(t)v_i(x,y)+\sum_{i=1}^{2}\sum_{j=}^{2}y_iy_jv_{ij} \qquad (2)$$

The functions $u_i,v_i,w_i,\psi_{x_i},\psi_{y_i}$ are the components of the eigen vector $\vec{U}_i$, and the functions u_{ij},v_{ij} must be solutions of the following system of the differential equations:

$$\begin{cases}L_{11}u_{ij}+L_{12}v_{ij}=-Nl_1^{(2)}(w_i,w_j)\\ L_{21}u_{ij}+L_{22}v_{ij}=-Nl_2^{(2)}(w_i,w_j)\end{cases} \qquad (3)$$

The right parts of the system (3), denoted as operators $Nl_k^{(2)}(w_i,w_j)$, $(k=1,2)$ have the following form:

$$Nl_1^{(2)}(w_i,w_j)=\frac{\partial w_i}{\partial x}\left(C_{11}\frac{\partial^2 w_j}{\partial x^2}+2C_{16}\frac{\partial^2 w_j}{\partial x\partial y}+C_{66}\frac{\partial^2 w_j}{\partial y^2}\right)+$$

$$+\frac{\partial w_i}{\partial y}\left(C_{16}\frac{\partial^2 w_j}{\partial x^2}+(C_{12}+C_{66})\frac{\partial^2 w_j}{\partial x\partial y}+C_{26}\frac{\partial^2 w_j}{\partial y^2}\right)),$$

$$Nl_2^{(2)}(w_i,w_j)=\frac{\partial w_i}{\partial x}\left(C_{16}\frac{\partial^2 w_j}{\partial x^2}+(C_{12}+C_{66})\frac{\partial^2 w_j}{\partial x\partial y}+C_{26}\frac{\partial^2 w_j}{\partial y^2}\right))+$$

$$+\frac{\partial w_i}{\partial y}\left(C_{66}\frac{\partial^2 w_j}{\partial x^2}+2C_{26}\frac{\partial^2 w_j}{\partial x\partial y}+C_{22}\frac{\partial^2 w_j}{\partial y^2}\right))$$

It should be noted that the system (3), added the corresponding boundary conditions and also the natural vibration problem were carried out by RFM method [5, 6].

Substituting the expressions (2) for unknown functions u, v, w, ψ_x, ψ_y into third equations of the system (1) and applying the procedure by Bubnov-Galerkin one can obtain nonlinear system of the ordinary differential equations in $y_1(t), y_2(t)$ of the following form:

$$y_1^{//} + \alpha_0^{(1)} y_1 + \alpha_{11}^{(1)} y_1^2 + \alpha_{12}^{(1)} y_1 y_2 + \alpha_{22}^{(1)}{}_{22} y_2^2 + \gamma_{111}^{(1)} y_1^3 + \gamma_{112}^{(1)} y_1^2 y_2 + \gamma_{122}^{(1)} y_1 y_2^2 + \gamma_{222}^{(1)} y_2^3 = \alpha_r P(t)$$

$$y_2^{//} + \alpha_0^{(2)} y_2 + \alpha_{11}^{(2)} y_1^2 + \alpha_{12}^{(2)} y_1 y_2 + \alpha_{22}^{(2)} y_2^2 + \gamma_{111}^{(2)} y_1^3 + \gamma_{112}^{(2)} y_1^2 y_2 + \gamma_{122}^{(2)} y_1 y_2^2 + \gamma_{222}^{(2)} y_2^3 = 0 \qquad (4)$$

The factors of the equations are defined by formulas:

$$\alpha_0^{(m)} = \omega_{mL}^2$$

$$\alpha_{ij}^{(m)} = -\frac{1}{m_1 \|w_m\|^2} \iint_\Omega \Big(k_1 N_{11p}{}^{(ND)}\big(u_{ij}, v_{ij}, w_i, w_j\big) + k_2 N_{22}{}^{(ND)}\big(u_{ij}, v_{ij}, w_i, w_j\big) + N_{11}^L(u_i, v_i, w_i) w_{i,xx} +$$

$$+ N_{22}^L(u_i, v_i, w_i) w_{i,yy} + 2N_{12}^L(u_i, v_i, w_i) w_{i,xy} \Big) d\Omega, \quad (m, i, j = 1,2)$$

$$\gamma_{ijk}^{(m)} = -\frac{1}{m_1 \|w\|^2} \iint_\Omega \Big(k_1 N_{11p}{}^{(ND)}\big(u_{ij}, v_{ij}, w_i, w_j\big) w_{k,xx} + N_{22}{}^{(ND)}\big(u_{ij}, v_{ij}, w_i, w_j\big) w_{k,yy} +$$

$$+ 2N_{12}^{(ND)}\big(u_{ij}, v_{ij}, w_i, w_j\big) w_{k,xx} \Big) w_m d\Omega$$

Let us denote the following expressions in matrix form:

$$(N_L)^T = \left(N_{11}^L, N_{22}^L, N_{12}^L\right), \quad (N_{NL})^T = (N_{11}^{(ND)}, N_{22}^{(ND)}, N_{12}^{(ND)})$$

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{pmatrix}, \quad E_L^{sh} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix} = \begin{pmatrix} u_{i,x} - k_1 w_1 \\ v_{i,y} - k_2 w_1 \\ v_{i,x} + u_{i,y} \end{pmatrix}, \quad E_L^{pl} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix} = \begin{pmatrix} u_{ij,x} \\ v_{ij,y} \\ v_{ij,x} + u_{ij,y} \end{pmatrix}$$

$$NDW_{ij} = \begin{pmatrix} w_{i,x}\, w_{j,x} \\ w_{i,y}\, w_{j,y} \\ w_{i,x}\, w_{j,y} + w_{i,y}\, w_{j,x} \end{pmatrix}$$

Then expressions for $N_{11p}^{(Nd)}, N_{22p}^{(Nd)}, N_{12p}^{(Nd)}, N_{11}^L, N_{22}^L, N_{12}^L$ may be defined as corresponding components of the following vectors: $N_L = CE_L^{sh}$, $N_{NL} = CE_L^{pl} + \frac{1}{2} NDW_{ij}$.

The obtained system (4) is solved by Runge-Kutta method and special stability criterion described below.

3. INVESTIGATION OF STABILITY OF NONLINEAR VIBRATION MODES

Let us consider the stability of the second periodic or chaotic vibration form which is defined as $y_2(t) = 0$. Instability of the form $y_2(t) = 0$ means "swap" of energy from one harmonic of the Fourier series in another one. The variables $y_2(t)$, $\dot{y}_2(t)$ may be considered as variations. That is why we assume that value of the variation y_2 is essentially smaller than the variable y_1 in zone of stability of the vibration form, $y_2(t) = 0$, as it is accepted in stability theory.

Limited criterion for finding instability zones of nonlinear vibration modes for considered system is applied [7]. It is assumed that initial value of the variable *y* is not however small variable and so the connection between constant ε and value of δ [7] is introduced. Let the variable *t* be varied from *0* to *T*. Then the following criterion of stability/instability is taken:

Instability of the vibration form $y_2(t) = 0$ is fixed if the following condition

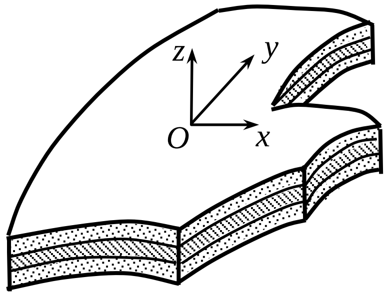
$$|y_2(t)| \geq \rho |y_2(0)|, \quad (0 \leq t \leq T)$$

holds true. The foregoing criterion, obtained provided that value δ can not be arbitrary small one is called as "limited criterion of stability" which is a consequence of the classical criterion of stability by Lyapunov [7,8]. Here the value ρ^{-1} is an order infinitesimal of the initial variation with respect to maximum permissible variation of ε for any $t \geq 0$. The increasing value ρ means that allowable initial variations are decreasing. It exists some arbitrariness while choosing ρ; it is not by accident because in the instability region the variations while increase t come out limits of the initial solution ε - neighborhood for any choose of parameter ρ. For definiteness this value is taken as $\rho \le 10$.

For determining the finite value of parameter T experiment calculation is realized into some mesh points at chosen scale of system parameters at the fixed value of T. Increasing the value of parameter T and corresponding calculations will be continued until boundaries of instability zones are stabilized at chosen scale of variables plane.

4. NUMERICAL RESULTS

Find the stability zones of the vibration forms for clamped three layers shallow spherical shells presented in Fig. 1(a) and supported on plan form shown in Fig 1(b).



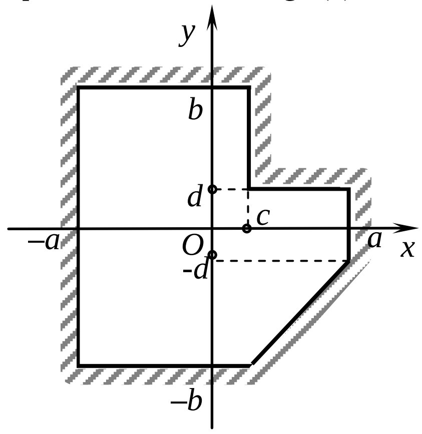


Fig.1(a,b). Plan of clamped three layers shallow shells

The shell is loaded transverse force, $P(t) = F\cos\Omega t$. The mechanical characteristics of the shell are: $E = 25E_2,\ G_{12} = G_{13} = 0.5E_2,\ G_{23} = 0.2E_2,\ \nu_{12} = 0.25$. The shear factors are taken to be $k_4^2 = k_5^2 = 5/6$. The geometric parameters of the shell are taken as follows:

$$b/a = 1,\quad c/(2a) = d/(2a) = 0.25,\quad 2a/R_x = 2a/R_y = 0.1,\quad h/(2a) = 0.1$$

The boundary conditions are accepted in the following form (clamped edge):

$$w = 0,\ u = 0,\ v = 0,\ \psi_x = \psi_y = 0 \quad \forall(x, y) \in \partial\Omega,$$

Here $\partial\Omega$ denotes the whole boundary of the domain, the equations of which may be constructed with help R-functions theory

$$\omega(x, y) = (f_1 \wedge_0 f_2) \wedge_0 (f_3 \vee_0 f_4) \wedge_0 f_5$$

The functions $f_i,\ (i = \overline{1,5})$ are defined as follows:

$$f_1 = (b^2 - y^2)/(2b) \ge 0,\quad f_2 = (a^2 - x^2)/(2a) \ge 0,\quad f_3 = (c - x) \ge 0,\quad f_4 = (d - y) \ge 0$$
$$f_5 = ((y + b)(a - c) - (x - c)(b - d)) \ge 0$$

The expressions for R-operators $\wedge_0, \vee_0$ are defined according to [5]. The corresponding structural formulas [5, 6] are

$$u = \omega P_1,\quad v = \omega P_2,\quad w = \omega P_3,\quad \psi_x = \omega P_4,\quad \psi_y = \omega P_5$$

Here $P_i, (i = 1,2,3,4,5)$ are indefinite components of the constructed structures of solution, which are expanded in series in some complete system of functions. The coefficients of this expansion are sought from the stationary condition for corresponding functional.

Values of the dimensionless parameter of the natural frequency for the three layers cross-ply $(0^0, 90^0, 0^0)$ spherical shell, panel and plate are presented in the Table 1.

Table 1. Values of the Dimensionless Frequency Parameter $\Lambda_i = \lambda_i \frac{(2a)}{h}\sqrt{\frac{\rho}{E_2}}$

$(2a/R_x,\ 2a/R_y)$	Λ_1	Λ_2	Λ_3	Λ_4
(0.1, 0.1)	18.851	29.139	36.113	43.113
(0, 0.1)	18.561	29.068	36.085	43.054
(0, 0)	18.453	29.035	36.069	43.031

Instability zones are presented for cross-ply spherical shells in the Fig. 2. Term of stabilization is *T=1000*, that is, calculation time, at which stabilization of boundaries of instability zones are observed.

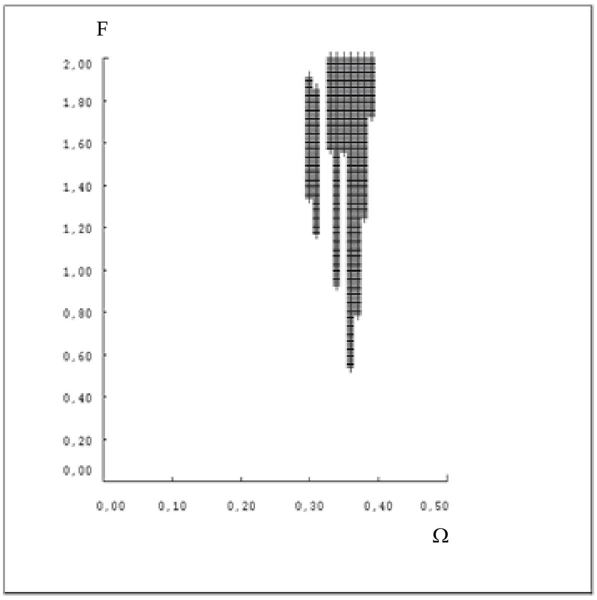


Fig.2. Instability zone for cross-ply spherical shell

Calculations were carried out in variable scale plane (Ω, F), where parameter Ω was varied $0 \le \Omega \le 0.5$ and parameter *F* was varied $0 \le F \le 2$.

Obtained results show that considered system may have instability behaviour at the parameter values starting with Ω *=0.27* and *F=0.42*.

CONCLUSION

The effective approach for investigation of stability of nonlinear vibration modes of the laminated shallow shells resting on complex plan form and having symmetric structure of layers are proposed. The method is based on R-functions theory, variational methods, special criterion of stability and method by Runge-Kutta. There is presented numerical results for clamped three layered spherical shallow shells.

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