INVESTIGATION OF STABILITY WITH RESPECT TO PART OF VARIABLES IN HYBRID AUTOMATA

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ABSTRACT

In the article new constructive conditions for stability of trivial equilibrium point of hybrid automaton with respect to part of variables are proposed. Conditions are based on construction of a sequence of values of Lyapunov functions at state switching points. Proposed conditions principally differ from existing conditions in that they do not depend on the values of hybrid automaton's solutions at moments of switching.

INTRODUCTION

Hybrid automaton is a tuple $H = (Q,\ X,\ F,\ Init,\ Inv,\ Jump, \tau)$, where

- τ is a hybrid time,
- $X = \{x_1, ..., x_n\}$, $n \geq 0$, $x_i \in R$ is a phase space;
- $F = \{f_i\ :\ Q \times R^n \to R^n,\ i = \overline{1,\ N}\}$ are right hand sides of differential equations which describe dynamics in local states;
- $Init$: $Init \subset Q \times R^n$ is a set of initial states;
- Inv : $Inv \subset Q \times R^n$ is an invariant set of each local state;
- $Jump$: $Q \times R^n \to \beta(Q \times R^n)$ is a map which describes automaton's transitions.

We describe now the usage of Lyapunov's second method for investigation of stability of equilibrium point of hybrid automata.

One refer to existing methods of investigation of stability of hybrid automata.

Suppose that dynamics in i-th local state is described by systems of differential equations $\dot{x}(t) = f_i(t, x(t))$, $i = 1, ..., N$. Most of methods require a set of Lyapunov functions $\{V_i,\ i = 1, ..., N\}$ to be defined.

Existing approaches require non-increasing of Lyapunov functions at switching points on values of hybrid automaton's trajectories:

1) R. DeCarlo, D. Liberzon, A. Morse [1,2]:

$$V_j(x(t_{j+1})) - V_i(x(t_{i+1})) \leq -\gamma \|x(t_{i+1})\|^2,\ \gamma > 0,$$

where $t_i < t_j, i < j$ are switching moment (asymptotic stability).

2) M. Branicky [3]:

$$V_i(x(t_{i,k})) \leq V_i(x(t_{i,k-1})),$$

where $t_{i,k}$ is a k-th moment of switching to the vector field f_i.

3) H. Ye, A. Michel [4] use the "weak Lyapunov function":

$$V_i(x(t)) \leq h(V_i(x(t_j))), t \in (t_j,\ t_{j+1}),$$

where $h : R^+ \to R^+$ is a continuous function which satisfies initial condition $h(0) = 0$, t_j is an arbitrary switching moment.

As noted earlier, proposed conditions depend on values of trajectory at switching moments.

1. HYBRID AUTOMATA STABILITY CONDITIONS

Let the space R^2 and two local states be given. The first local state is defined by subset of phase space,

$$\Omega_1 = \{x \in R^2 : x_2 - kx_1 < 0\},$$

the second one is defined as,

$$\Omega_2 = \{x \in R^2 : x_2 - kx_1 > 0\}.$$

Thereby transition from the state I to the state II occurs when a trajectory reaches the line $x_2 = kx_1$. Suppose that in local states dynamics is described by systems of linear differential equations.

Assume that for each system there exists a positive-definite Lyapunov function such that

$$\left.\frac{dV_i(x)}{dt}\right|_{(i)} < 0\text{, if } \quad x \in \Omega_i\text{, } i = 1,2. \tag{1}$$

Thus we require (1) only on the set which defines current local state.

Choose an arbitrary point x^0 on the switching line. Let I be an initial state. Let us build a level set of the function $V_1(x)$ which starts at x^0 (fig. 1).

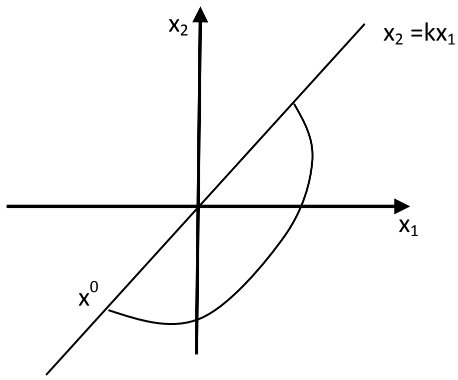

Fig. 1. The first Lyapunov's function level-line

Denote $c^0 = V_1(x^0)$. Let us find an intersection point of a level set $C_1 = \{x \in R^2 : V_1(x) = c^0, x \in \Omega_1\}$ with the line $x_2 = kx_1$. Denote it by x^1 (fig. 1).

At the point x^1 the switching occurs from the state I to the state II. Therefore let $c^1 = V_2(x^1)$. Let us build a level set of the Lyapunov function $V_2(x)$, which starts at x^1 in the second local state $C_1 = \{x \in R^2 : V_1(x) = c_0, x \in \Omega_1\}$ (fig. 2).

Let us find an intersection point of the level set $C_1 = \{x \in R^2 : V_1(x) = c_0, x \in \Omega_1\}$ with the line $x_2 = kx_1$. Denote it by x^2 (fig. 2).

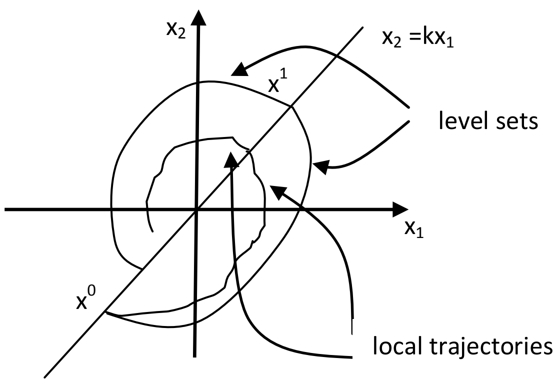

Fig. 2. The R^2 Lyapunov's function level-line and trajectories

From the inequality $\left.\frac{dV_1(x)}{dt}\right|_{(1)} < 0$ in the first local state it follows that a trajectory, which starts at x^0 can not leave a domain bounded by the level set $C_1 = \{x \in R^2 : V_1(x) = c^0, x \in \Omega_1\}$. Analogously, trajectory of the second local state, which starts at x^1 can not leave a domain bounded by the level set $C_1 = \{x \in R^2 : V_1(x) = c_0, x \in \Omega_1\}$ (fig. 2).

Therefore it can be assumed that if

$$\left|x^2\right| < \left|x^0\right|, \tag{2}$$

then the trivial equilibrium point of the hybrid automaton is asymptotically stable.

If should be noted that condition (2) can not be applied in cases when transition occurs on non-straight lines. For example, suppose that transition occurs on the curve shown on the fig. 3.

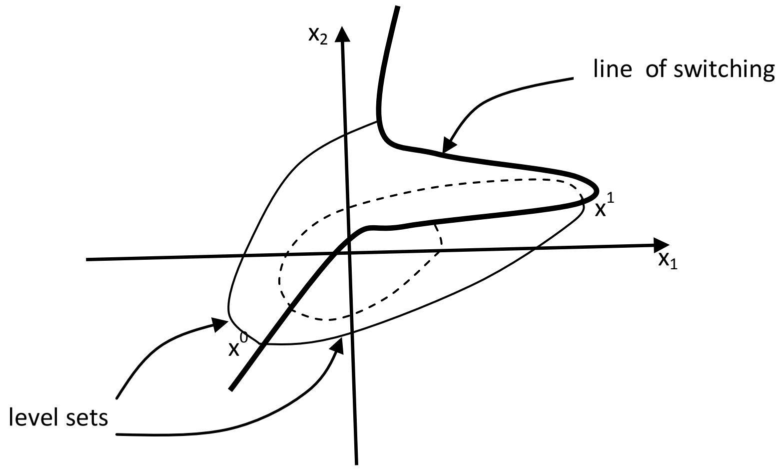

Fig. 3. The R^2 Lyapunov's function level-line and trajectories with non-straight line of switching

It is obvious that condition (2) is not satisfied but trivial equilibrium can be stable. Therefore it is reasonable to use the following condition instead of (2)

$$\left|c_2\right| < \left|c_0\right| .$$

Condition (2) can be generalized to the cases when phase space is R^n and switching surfaces are arbitrary (fig. 4).

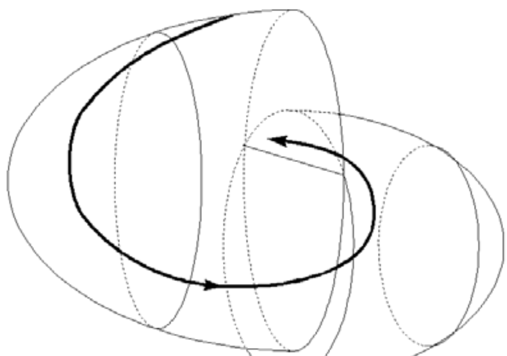

Fig. 4. The R^3 Lyapunov's function level-line and trajectories

Suppose that trajectory of hybrid automaton starts in the first state. We use the notation $x|_{i \to i+1}$ to indicate that hybrid automaton switches from state i to state $i+1$ at the point x.

To obtain constructive stability conditions let us build the following sequence (s-condition):

$$c^0 \in (0, C), \quad c^1 = \max_{\substack{x^1|_{1\to 2} \\ V^1(x^1) \le c_0}} V^2(x^1), \quad c^2 = \max_{\substack{x^2|_{2\to 3} \\ V^2(x^2) \le c_1}} V^3(x^2), \ldots, \; c^N = \max_{\substack{x^N|_{N\to 1} \\ V^N(x^N) \le c_{N-1}}} V^1(x^N) \tag{3}$$

Here we take into account the case when a level set intersects with a switching surface before (in time), a trajectory reaches switching surface.

In the condition (3) the second restriction allows to take into account a value of Lyapunov function at the switching point and to use it for construction of the next level set.

2. STABILITY WITH RESPECT TO PART OF VARIABLES IN HYBRID AUTOMATA

Let us build the following sequence $\{c^i\}$, $i=\overline{0,\,N}$:

$$c^0 \in (0,C), \qquad c_k(h) = \sup\{V_{\overline{k+1}}(x) \mid x \in J_k, V_k(x) \le h\} \tag{4}$$

It can be built starting from arbitrary initial state (not only from the first state). We introduce the following notation: $B_r = \{x \in R^n \;:\; |x| \le r\}$, $S_r = \{x \in R^n \;:\; |x| = r\}$.

We define a hybrid time τ either as a finite sequence $\tau_H = \{\tau_i\}_1^N$, where: $\tau_i = (Pre\_jump_i,\ [t^*_{i-1}, t^*_i],\ Post\_jump_i), i = 1..N;$ $t_0 = 0$; $[t^*_{i-1}, t^*_i]$ are closed segments, and the last element is a semi-open interval $\tau_N = [t_N, \infty)$, or as an (infinite) sequence $\tau_H = \{\tau_i\}_1^\infty$ of closed segments $[t^*_{i-1}, t^*_i]$. Denote T is the set of all possible τ.

Definition 1. A phase orbit of hybrid automaton H is a set $\chi = \{(\tau, i, x)\}$, where $\tau \in T$, i – a number of a local state and $x \;:\; \tau \to R^n$ is a function such that $(i(\tau_0),\ x(\tau_0)) \in Init,$ for all u such that $\tau_i < \tau'_i$. Here $(i(t),\ x(t)) \in Inv$ defines continuous dynamics in i-th local state and $(i(\tau_{i+1}),\ x(\tau_{i+1})) \in Jump(i(\tau'_i),\ x(\tau'_i))$ defines discrete dynamics.

Definition 2. Continuous state $x = 0$ is called a trivial equilibrium point of hybrid automaton if (i) there exists a non-empty set $\overline{Q} \subset Q$, such that for all $i \in \overline{Q}$ condition $(i',\ z') \in Jump(i,\ 0)$ implies that $z' = 0$ and $i' \in \overline{Q}$; (ii) $f(i,\ 0) = 0$ for all $i \in Q$.

Definition 3. Trivial equilibrium of hybrid automaton H is called stable (in sense of Lyapunov), if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every trajectory the condition $|x(t_0)| < \delta$ implies that $|x(t)| < \varepsilon$ for all $t \in \tau$. Here $|\cdot|$ denotes Euclidean norm.

Let us denote Ω_i the set which describes i-th local state.

Assume that there exist Lyapunov functions defined on the sets Ω_i.

Definition 4. An indexed family $V(i,\ x) = \{V^i(x)\},\ i = \overline{1,\ N}$ is called a hybrid s-function, if each $V^i(x)$ is positive definite and for every sequence $\{c^i\},\ i = \overline{0,\ N}$ defined as in (4) the inequality $c^N \le c^0$ holds.

We will use hybrid s-function for investigation of stability of trivial equilibrium point of hybrid automata.

Definition 5. The following expression is called a derivative of hybrid s-function with respect to hybrid automaton:

$$\dot{V}(i,x) = \left\{ \frac{dV^i(x)}{dx} f_i(x(t)),\ i = \overline{1,\ N} \right\}.$$

Theorem 1. [5] Suppose that hybrid automaton H has a trivial equilibrium point, $|Q| < \infty$, $i = \overline{1, N-1},\ Jump(N, x) = (1, x)$. Also suppose that a neighborhood of the coordinate origin $D \subset X$ is given. If there exists a positive-definite hybrid s-function $V(i,\ x)$: $Q \times D \to R$ for hybrid automaton H, such that $\frac{dV^i(x)}{dx} f_i(\,x(t)) \le 0$ for all $x \in D \cap \Omega_i$ and $i = \overline{1,\ N}$, then $x = 0$ is stable trivial equilibrium point of hybrid automaton H.

It should be noted that checking of the proposed condition does not require investigation of reachability of switching surface by hybrid automaton's trajectories. It is connected with the fact that if switching surface is not reachable and s-condition is satisfied, then stability of equilibrium follows from classical Lyapunov theorem, because in this case we can simply consider system on the whole phase space.

Also a principal value has the fact that proposed theorem does not require computation of hybrid automaton's solution.

Let us construct stability conditions from impulsive hybrid automata.

Suppose that hybrid automaton's trajectory starts in the first state. We use the notation $x^{i-} \to x^{i+} |_{i \to i+1}$ to indicate that hybrid automaton switches from the state i to $i+1$ and the value x^{i-} is taken from the set which determines jump condition, while x^{i+} is the value of phase coordinate after jump, i.e. $x^{i+} = q(x^{i-})$.

Theorem 2. Suppose that hybrid automaton H has the trivial equilibrium point $x = 0$, $|Q| < \infty$, $Jump(i, x) = \{(i+1, q_i(x))\}$, for $i = \overline{1, N-1}$, $Jump(N, x) = (1, q_N(x))$. Also suppose that a neighborhood of the origin $D \subset X$ is given. If for each local state Ω_i there exist positive-definite functions $V(i, x)$: $Q \times D \to R$ such that

1. $\frac{dV^i(x)}{dx} f_i(x(t)) \le 0$ for all $x \in D \cap \Omega_i$, $i = \overline{1, N}$;
2. for every sequence c^i which starts in arbitrary state the condition $c^N \le c^0$ is satisfied;
3. there exists a continuous monotone increasing function $\psi(\cdot): R^+ \to R^+$, such that $\psi(0) = 0$

and $\|q_i(z)\| < \psi(\|z\|)$, $\forall i \in Q$. Then $x = 0$ is stable trivial equilibrium point of impulsive hybrid automaton *H*.

Also corresponding theorems about instability and exponential stability have been proved.

Now let us turn to the problem of stability with respect to part of variables. Consider a hybrid automaton described by the equations of the following kind in it's local states:

$$\dot{y} = f_q(y),\ y \in Inv_q,\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ y(t_0) = 0, \quad (5)$$

where $y_1 \in R^{n_1}$, $y_2 \in R^{n_2}$. We call a variable y_1 as observable, y_2 as hidden. We assume that transitions between states $q \in Q$ are continuous ($Jump(q, y) = \varnothing \vee \{(r, y)\}$).

A problem: determine stability of hybrid automaton (5) with respect to vector of observable coordinates y_1. We assume that $y_1 = 0$ is a trivial equilibrium point of hybrid automaton for each values of hidden vector y_2.

Denote $n = n_1 + n_2$, $|x|$ as Euclidean norm in 0_1, $|x|_1$ and $|x|_2$ as Euclidean norms in R^{n_1} and R^{n_2} correspondingly. Similarly, 0_1 denotes null-vector in R^{n_1}, 0 is a null-vector in R^n.

Definition 5. A trajectory $y(\bar{y}^0, t)$ of dynamical system $y(y^0, t)$ is called stable with respect to variables y_1, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that inequality $\|y^0 - \bar{y}^0\| < \delta$ implies $\|y_1(\bar{y}^0, t) - y_1(y^0, t)\|_1 < \varepsilon$.

Partial case 1. Suppose that switching in automaton (5) occurs only with respect to hidden coordinates y_2. If for the system (1) in some neighborhood $y_1 \in B_r(0_1)$ there exists a Lyapunov's function $V(y_1)$ such that $\dot{V}|_{f_q} \le 0$, then solution is stable.

Partial case 2. Suppose that for automaton (5) there exists a set of y_1-positive definite Lyapunov functions such that $\dot{V}^q|_{f_q} \le 0$, and on switching $y|_{q \to r}$ the inequality $V^r(y) \le V^q(y)$ holds. Then trivial equilibrium point of hybrid automaton is stable.

Let us mention a theorem about stability in general case.

Definition 6. A function $V(y): B_r(0_1) \times R^{n2} \to R$ is called y_1-uniform-positive-definite if there exist two positive definite functions $W(y_1), U(y_1): B_r(0_1) \to R$ such that for each $y = (y_1, y_2) \in B_r(0_1) \times R^{n_2}$ the inequality $W(y_1) \le V(y_1, y_2) \le U(y_1)$ holds.

Theorem 3. Suppose that hybrid automaton has cyclic continuous switching. If for a cylinder $D \times R^{n_2}$, where $D \subseteq R^{n_1}$, there exists a set of y_1-uniform-positive-definite Lyapunov functions $V^q: D \times R^{n_2} \to R$ such that $\dot{V}^q|_{f_q} \le 0$ for all $y \in D \times Inv_q$ and $c^N \le c^0$, then $x = 0$ is a stable trivial equilibrium point.

CONCLUSIONS

In the paper the constructive conditions for stability of trivial equilibrium point of hybrid automaton are proposed. Conditions are based on existence of hybrid s-functions and they do not depend on solutions as in classical Lyapunov theory. Obtained conditions are extended to impulsive hybrid automata. For investigation stability with respect to part of variables of hybrid automata a notion of y_1-uniform-positive-definite function is introduced, where y_1 is a phase subspace vector analyzed for stability.

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