Proceedings of the 3rd International Conference on Nonlinear Dynamics
ND-KhPI2010
September 21-24, 2010, Kharkov, Ukraine

NONLINEAR NORMAL MODES IN PENDULUM SYSTEMS.

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ABSTRACT

Nonlinear normal modes in some pendulum systems and a stability of these modes are analyzed. Namely, dynamics of the spring pendulum and of the 2-DOF system, containing a linear oscillator and the attached pendulum, is considered. Nonlinear normal modes are obtained as by the multiple scales method, as well by construction of trajectories in configuration space. Stability of nonlinear normal modes is investigated by using the Mathieu and Hill equations, and by the algebraization of the equations in variations. Numerical simulation confirms an exactness of obtained analytical results.

INTRODUCTION

Pendulum systems are classical models in mechanics and theory of nonlinear vibrations. Their analysis permits to select important nonlinear dynamical effects [1,2]. Besides, such systems are used in engineering, in particular, in the absorption problems [3,4], and to describe some physical processes [5,6]. In spite of numerous investigations of the pendulum dynamics, as in the past [7], as well at present [8,9], analytical results are obtained only for vibrations having not large amplitudes. In this work new asymptotical methods and numerical simulations are used to construct nonlinear normal modes and analyze their stability. Dynamics of the spring pendulum and of the 2-DOF system, containing a linear oscillator and the attached pendulum, is considered as for small, as well for large vibration amplitudes. Stability of the nonlinear normal modes is investigated too.

1. PENDULUM SYSTEMS UNDER CONSIDERATION

A model of the spring pendulum is shown in the Fig.1. Free vibrations of the system is described by two generalized coordinates, ρ and φ. Dissipation forces are not taken into account.

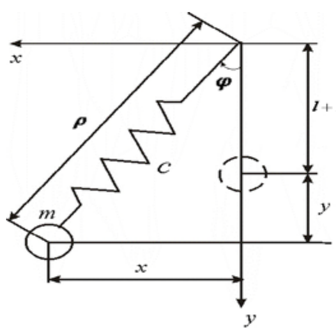

Fig. 1. The spring pendulum

Equations of motion are the following:

$$\begin{cases} \ddot{\rho} - \rho\dot{\varphi}^2 = -\dfrac{c}{m}(\rho - l) + g\cos\varphi; \\ \rho^2\ddot{\varphi} + 2\rho\dot{\rho}\dot{\varphi} = -g\rho\sin\varphi. \end{cases} \quad (1)$$

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Terms of the power more than three by φ in Taylor expansions of the functions $\cos\varphi$ and $\sin\varphi$, are discarded. One has the next transformation for a case of small values of the angle and spring dilation: $\varphi \to \mu\varphi$, $\rho - \rho_0 \to \mu z$, where μ is a small parameter. Then the equations of motion can be rewritten as

$$\begin{aligned}&\ddot{z} + z = \mu(\rho_0\dot{\varphi}^2 + 0.5g\varphi^2) + \mu^2\dot{\varphi}^2 z;\\&\rho_0^2\ddot{\varphi} + g\rho_0\varphi = \mu(-2\rho_0\ddot{\varphi} - 2\rho_0\dot{\varphi}\dot{z} - gz\varphi) + \mu^2(-z^2\ddot{\varphi} - 2z\dot{z}\dot{\varphi} + g\rho_0\varphi^3/6) + \mu^3 gz\varphi^3/6\end{aligned} \qquad (2)$$

where $\rho_0 = l - \dfrac{gm}{c}$ is the spring extension in the system equilibrium position.

It is possible to select two next vibration modes in the system: a) longitudinal vibrations, when the rotation is absent, $\varphi = 0$, $z = z(t)$; b) coupled vibrations, $\varphi = \varphi(t)$, $z = z(t)$.

The other two-DOF system is shown in the Fig. 2. A pendulum in the system can be considered as absorber of linear vibrations of the main linear oscillator. Vibrations of the system is described by two generalized coordinates, x and θ.

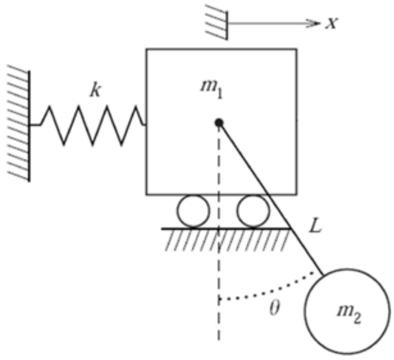


Fig. 2. The two-DOF system containing the pendulum as absorber.

Equations of motion are here the following:

$$\begin{cases}(m_1 + m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta + kx = 0;\\ \ddot{x}\cos\theta + l\ddot{\theta} + g\sin\theta = 0.\end{cases} \qquad (3)$$

Using the Taylor expansions for functions $\cos\varphi$ and $\sin\varphi$, we save only terms of the power not more than three by φ. One assumes that the mass of the pendulum is essentially smaller than one of the linear subsystem. Using the next transformation, $m_2 \to \varepsilon m_2$, where ε is a formal small parameter, it is possible to obtain equations of motion of the form:

$$\begin{cases}(m_1 + \varepsilon m_2)\ddot{x} + \varepsilon m_2 l\ddot{\theta}\left(1 - \dfrac{\theta^2}{2}\right) - \varepsilon m_2 l\dot{\theta}^2\left(\theta - \dfrac{\theta^3}{6}\right) + kx = 0;\\ \ddot{x}\left(1 - \dfrac{\theta^2}{2}\right) + l\ddot{\theta} + g\left(\theta - \dfrac{\theta^3}{6}\right) = 0.\end{cases} \qquad (4)$$

Two vibration modes exist in the system, namely: a) coupled vibrations, $x = x(t)$, $\theta = \theta(t)$; b) localized vibration mode, when values of vibration amplitude of the pendulum are essentially large than ones of the linear subsystems of the mass m_1.

2. NONLINEAR NORMAL MODES IN PENDULUM SYSTEMS

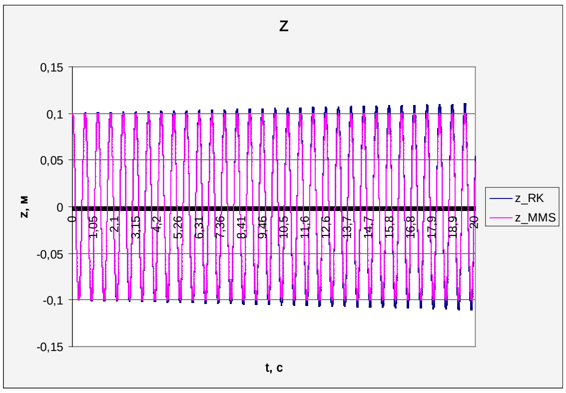
2.1. Construction of nonlinear normal modes for small amplitudes

To construct the mode of coupled vibrations for the system (2) the multiple scales method is used. Namely, series by the small parameter: $z = z_0 + \mu z_1 + \mu^2 z_2 + \ldots$, $\varphi = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \ldots$, and the presentations $\begin{matrix} z(t,\mu) = z(T_0, T_1, T_2, \ldots; \mu); \\ \varphi(t,\mu) = \varphi(T_0, T_1, T_2, \ldots; \mu), \end{matrix}$ where $T_0 = t$, $T_1 = \mu t$, $T_2 = \mu^2 t, \ldots$, are used. The next transformations to construct the periodic solution are not presented here. As a result, on has the periodic solution of the system (2):

$$z_1 = A_1 \cos \omega_1 T_0 + B_1 \sin \omega_1 T_0 + \frac{g}{4}\left(C_0^2 + D_0^2\right) +$$
$$+\frac{g}{1-4\omega_2^2}\left(-\frac{3}{2}C_0 D_0 \sin 2\omega_2 T_0 + \frac{3}{4}\left(D_0^2 - C_0^2\right)\cos 2\omega_2 T_0\right);$$
$$\varphi_1 = C_1 \cos \omega_2 T_0 + D_1 \sin \omega_2 T_0 +$$
$$+\frac{\omega_2\left(\omega_1 + \frac{\omega_2}{2}\right)}{g-\rho_0\left(\omega_1+\omega_2\right)^2}\left(\left(A_0 C_0 - B_0 D_0\right)\cos\left(\omega_1+\omega_2\right)T_0 + \left(B_0 C_0 + A_0 D_0\right)\sin\left(\omega_1+\omega_2\right)T_0\right)+$$
$$\frac{\omega_2\left(\omega_1 - \frac{\omega_2}{2}\right)}{g-\rho_0\left(\omega_1-\omega_2\right)^2}\left(\left(B_0 D_0 - A_0 C_0\right)\cos\left(\omega_1-\omega_2\right)T_0 + \left(A_0 D_0 - B_0 C_0\right)\sin\left(\omega_1-\omega_2\right)T_0\right). \tag{5}$$

where $\omega_1 = \sqrt{c/m}$, $\omega_2 = \sqrt{g/m}$, expressions of the functions $A_0 = A_0\left(T_2, T_3, ...\right)$, $B_0 = B_0\left(T_2, T_3, ...\right)$, $C_0 = C_0\left(T_2, T_3, ...\right)$, $D_0 = D_0\left(T_2, T_3, ...\right)$ are not presented here. One has the very good coincidence of the analytical results and numerical simulation by the Runge-Kutta method for nor large vibration amplitudes.

In the Fig.3 it is presented a comparison of the analytical results and numerical simulation by the Runge-Kutta method for nor large vibration amplitudes. In the Fig.3,a it is shown a change in time of the variation z, and in the Fig. 3,б it is shown a change of the variation φ.



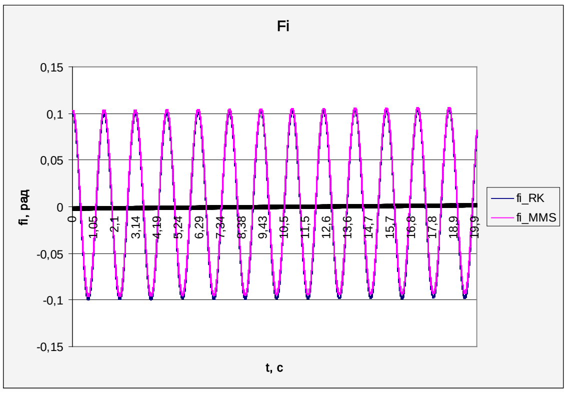


Fig. 3,a *Fig. 3,б*

Fig.3. Mode of the coupled vibrations obtained by the method of multiple scales and by the numerical simulation.

The same approach is used to construct vibrations of the system (3). A good correspondence of the analytical and numerical results is obtained.

2.2. Construction of nonlinear normal modes for large amplitudes

To construct coupled vibrations with large amplitudes theory of nonlinear normal modes (NNMs) is used [10-12]. Equation for trajectories of motions $z = z(\varphi)$ for the system (2) may be obtained of the form:

$$\left(h-V\right)m\left(\left(\mu z+\rho_0\right)\left(z''-\left(z+\rho_0\right)\right)-2\mu z'^2\right)+$$
$$+\tilde{K}\left(\left(\mu z+\rho_0\right)\left(c\left(z+\rho_0-l\right)-mg\left(1-\frac{\varphi^2}{2}\right)\right)-mz'g\varphi\right)=0 \tag{6}$$

where prime means a derivation by φ; V and $\tilde{K}$ are respectively the system potential energy and the system kinetic energy. Equation (6) has singular points on the maximal equipotential surface, $h - V = 0$. Additional boundary conditions guarantee an analytical continuation of the NNM trajectory to this surface [10-12]:

$$-mz'g\varphi\tilde{K} + c\tilde{K}(z+\rho_0-l)(\mu z+\rho_0) - mg\tilde{K}(\mu z+\rho_0)(1-\varphi^2/2) = 0, \tag{7}$$

where $\varphi = \varphi_0$, and $\dot{\varphi} = 0$. Solution of the boundary problem (6) and (7) can be obtained in power series by φ. Amplitudes values $\varphi = \varphi_0$ depending on the energy value $h = h_0$ are obtained too.

Numerical simulation shows a very good exactness of the analytical solution for large vibration amplitudes (Fig.4).

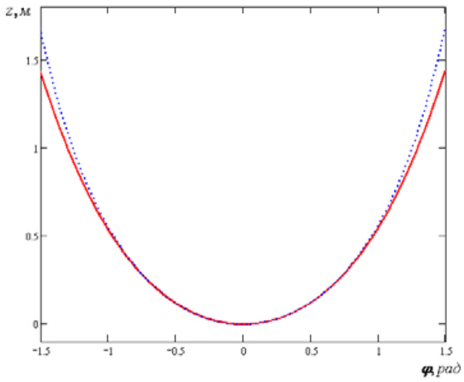


Fig.4. Trajectory of mode of coupled vibrations in configuration space. Entire line represents the analytical solution; point line represents the checking numerical calculation.

Construction of the mode of coupled vibrations in the power series by $\cos\varphi$,

$$z(\varphi) = z_0 + \mu z_1 = \alpha_0 + \alpha_1 \cos\varphi + \alpha_2 \cos^2\varphi + \mu\left(\beta_0 + \beta_1 \cos\varphi + \beta_2 \cos^2\varphi\right) + \ldots, \tag{8}$$

is very effective for large vibrations too.

Nonlinear normal modes of the system (3) are determined by construction of their trajectories in a configuration space too. Equations and boundary conditions similar to ones (6), (7), are used. The power series are used for the NNMs construction. In Fig. 5 trajectories of the NNMs are presented. The non-localized mode of coupled vibrations, obtained in the form $x = x(\theta)$ is shown in Fig 5.a, and the localized mode, determined in the form $\theta = \theta(x)$, is shown in the Fig. 5b.

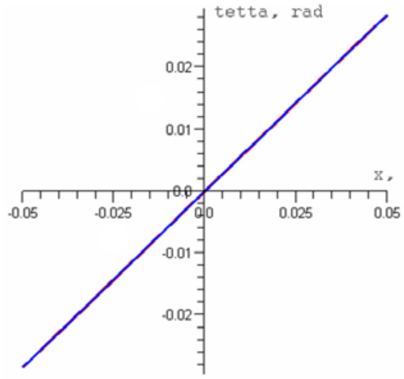


Fig.5a.

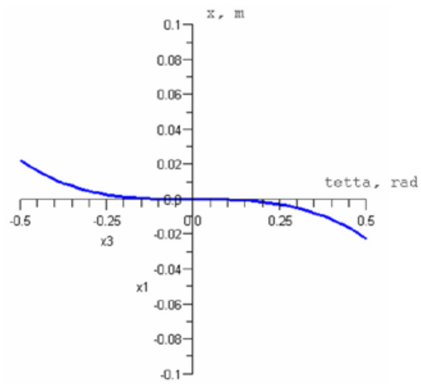


Fig 5.b.

Fig.5. Trajectories of mode of coupled vibrations in the system (3) configuration space. Fig.5a. Trajectory of the mode of coupled vibrations; Fig.5b. Trajectory of localized mode.

3. STABILITY OF NONLINEAR NORMAL MODES IN PENDULUM SYSTEMS

Stability of longitudinal motions investigated in details by many authors. The equation in variations, which are orthogonal to the rectilinear trajectory of the longitudinal vibration mode, is considered. The stability analysis may be made by reduction of the equation in variations to the Mathieu equation, or by the method of Hill determinants. In the last variant results are very close to ones obtained by the checking numerical simulation.

Stability of mode of the coupled vibrations are investigated by approach which is connected with the well known classical definition of stability by Lyapunov [13,14]. In this case the values of variables are compared with their initial values. Necessary condition of stability of motion is the following:

$$\sqrt{\left(\Delta z(t)\right)^2 + \left(\Delta\varphi(t)\right)^2} \le \xi \sqrt{\left(\Delta z(0)\right)^2 + \left(\Delta\varphi(0)\right)^2}, \tag{9}$$

where $\Delta z(t) = z(t) - z_0$, $\Delta\varphi(t) = \varphi(t) - \varphi_0$, $\Delta z(0) = z_0 / k$, $\Delta\varphi(0) = \varphi_0 / k$. Here z_0 и φ_0

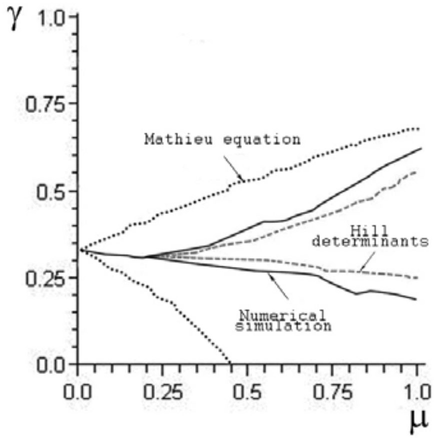


Fig.6. Boundaries of the stability/instability regions of the longitudinal vibrations.

are initial values of the corresponding variables. It exists some arbitrariness in choosing of the constants ξ, k. It is used that $\xi = 10, k = 100$. Violation of the condition (9) shows to instability of the solution. Numerical calculation is made in points of some mesh on a plane of the system parameters until boundaries of the stability/instability regions on this plane will be stabilized. These boundaries are shawn in Fig.7 on the plane of parameters $\omega^2 = c\rho_0 / mg$ and A which is the angle vibrations amplitude. The instability region is inside of the lines.

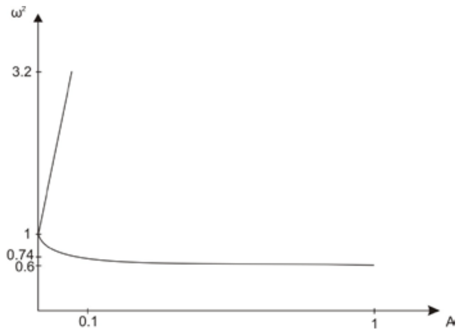


Fig.7. Boundaries of the stability/instability regions for mode of coupled vibrations.

The pairs of solutions fork from the mode of coupled vibrations in bifurcation points which correspond to the stability/instability boundaries. Examples of trajectories of these forking solutions are shown in Fig. 7 for different values of the angle amplitude.

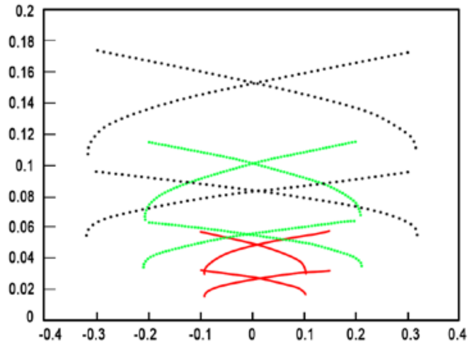


Fig.8. Trajectories of forking solutions in the pendulum configuration space.

In the Fig.9 it is presented boundaries of the stability/instability regions for the mode of coupled vibrations for the system (3), obtained by using the reduction to the Mathieu equation (exterior lines), and by using the more exact method of the Hill determinants (interior lines). Unstable vibrations are observed inside the lines. The forking solutions are shown for some values of the system parameters in the Fig. 10.

Stability of the localized vibration mode is investigated by the Hill determinants for the equation in variations. It is obtained that regions of the mode instability are very narrow.

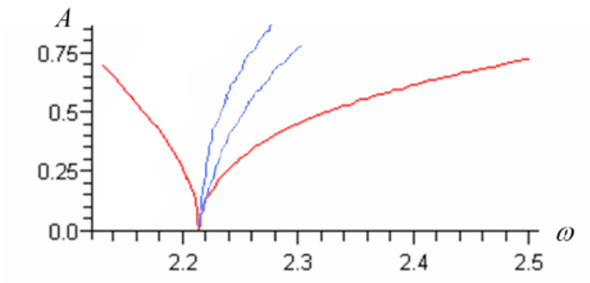


Fig.9. Boundaries of the stability/instability regions of the non-localized mode of the system (3)

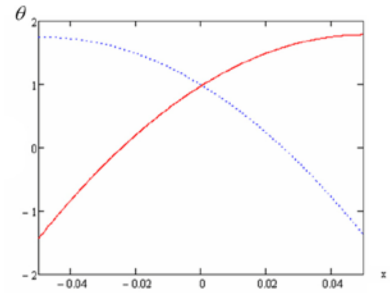


Fig.10. Trajectories of the forking solutions for the non-localized vibration mode.

CONCLUSIONS

The nonlinear normal modes in pendulum systems and their stability are investigated both for small, and for large vibration amplitudes Numerical simulation shows a good exactness of the obtained analytical results.

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