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ON VISUALIZATION OF RESONANCE STRUCTURES IN DYNAMICAL SYSTEMS WITH TWO DEGREES OF FREEDOM

T.N. Dragunov[1] ,
R.E. Kondrashov ,
A.D. Morozov
University of Nizhniy Novgorod
Nizhniy Novgorod, Russia

ABSTRACT

Resonance zones in a system of two weakly connected Duffinng - van der Pol nonlinear oscillators are analyzed by calculation of three-dimensional averaged system. Numerical visualization is suggested by using a two-dimensional Poincare map which is similar to the Poincare map for a system with 3/2 degrees of freedom. Visualization of resonance zones is performed using a computer program WInSet developed by authors.

INTRODUCTION

General essentially nonlinear systems close to integrable ones were considered in a book [1]. But papers concerning particular systems of two connected nonlinear oscillators and containing analysis of resonance structures have been presented only recently [2]. This may be connected with complexity of finding solution of unperturbed oscillator equation. In simplest cases the solution can be expressed in elliptic functions which significantly complicate an analysis of averaged systems. On the other hand, frequency of unperturbed oscillator depends on value of the energy integral and this fact leads to existence of dense set of its resonant values.

1. RESONANCE ZONES

A system of two weakly connected oscillators may be conveniently rewritten in variables of action I and angle ϑ: $I = (I_1, I_2),\ \vartheta = (\vartheta_1, \vartheta_2)$.

$$\begin{aligned} \dot{I} &= \varepsilon F(I, \vartheta) \\ \dot{\vartheta} &= \omega(I) + \varepsilon G(I, \vartheta) \end{aligned} \tag{1}$$

where $\omega = (\omega_1, \omega_2)$ and $\omega_k = \omega_k(I_k),\ k = 1{,}2$, ε is a small positive parameter, functions $F = (F_1, F_2), G = (G_1, G_2)$ are sufficiently smooth in the domain $D(I_1, I_2) \times T^2(\vartheta_1, \vartheta_2)$, where T^2 is a two-dimensional torus and $D \subset \mathbf{R}^2$.

It is said that there is a resonance in the system (1) if

$$\omega_1(I_1) = \frac{q}{p}\omega_2(I_2), \tag{2}$$

where p, q are relatively prime integers. Relation (2) defines resonance curves on the plane (I_1, I_2). Let us fix certain point (I_{1pq}, I_{2pq}) on a resonance curve. Inside the $\sqrt{\varepsilon}$-neighborhood of this point the system (1) can be reduced as in [1] by averaging and neglecting the terms $O(\varepsilon^{3/2})$ to the form

[1] Corresponding author. Email dtn@mm.unn.ru

$$
\begin{aligned}
u_k' &= A_k(v; I_{1pq}, I_{2pq}) + \mu[P_{k1}u_1 + P_{k2}u_2] \\
v' &= b_{10}u_1 + b_{20}u_2 + \mu[b_{11}u_1^2 + b_{21}u_2^2 + G_0(v; I_{1pq}, I_{2pq})]
\end{aligned} \tag{3}
$$

where $\mu = \sqrt{\varepsilon}$, the prime denotes a derivation by the "slow" time $\tau = \mu t$,

$$A_k = \frac{1}{2\pi p}\int_0^{2\pi p} F_k(I_{1pq}, I_{2pq}, v - q\varphi / p, \varphi)d\varphi$$

$$P_{k1} = \frac{1}{2\pi p}\int_0^{2\pi p} \frac{\partial F_k(\cdot)}{\partial I_1}d\varphi, \quad P_{k2} = \frac{1}{2\pi p}\int_0^{2\pi p} \frac{\partial F_k(\cdot)}{\partial I_2}d\varphi$$

$$G_0 = \frac{1}{2\pi p}\int_0^{2\pi p}[G_1(I_{1pq}, I_{2pq}, v - q\varphi / p, \varphi) + qG_2(I_{1pq}, I_{2pq}, v - q\varphi / p, \varphi) / p]d\varphi$$

$$b_{1,j-1} = \frac{d\omega_1(I_{1pq})}{jdI_1^j}, \quad b_{2,j-1} = \frac{q}{p}\frac{d\omega_2(I_{2pq})}{jdI_2^j}, \quad j = 1,2.$$

Functions A_k are periodic by v. Concluding the averaged system (3) we neglected terms $O(\mu^3)$ which depend on all variables u_1, u_2, and φ. Using the change of variables

$$u_2 = (w - b_{10}u_1 - \mu Q_0(v, I_{1pq}, I_{2pq})) / b_{20}$$

$$u_1 = u ,$$

we can transform the equations (3) to a more convenient form for analysis

$$
\begin{aligned}
v' &= w + \mu[a_{20}u^2 + a_{02}w^2 + a_{11}uw] \\
w' &= A(v; I_{1pq}, I_{2pq}) + \mu[C_1(v; I_{1pq}, I_{2pq})u + C_2(v; I_{1pq}, I_{2pq})w] \ , \\
u' &= A_1(v; I_{1pq}, I_{2pq}) + \mu[C_3(v; I_{1pq}, I_{2pq})u + C_4(v; I_{1pq}, I_{2pq})w]
\end{aligned} \tag{4}
$$

where

$$A = b_{10}A_1 + b_{20}A_2$$

$$a_{20} = b_{11} + b_{21}b_{10}^2 / b_{20}^2, a_{02} = b_{21} / b_{20}, a_{11} = -2b_{21}b_{10} / b_{20}^2$$

$$C_1 = b_{10}P_{11} - (b_{10}^2 / b_{20})P_{12} + b_{20}P_{21} - b_{10}P_{22}$$

$$C_2 = (b_{10} / b_{20})P_{12} + P_{22} + Q_0'$$

$$C_3 = P_{11} - (b_{10} / b_{20})P_{12}$$

$$C_4 = P_{11} / b_{20}$$

2. MODEL SYSTEM

One considers a particular system of two connected Duffing-van der Pol equations

$$
\begin{aligned}
\ddot{x} + \alpha x + \beta x^3 &= \varepsilon[(p_1 - x^2)\dot{x} + p_2 y] \\
\ddot{y} + \gamma y + \delta y^3 &= \varepsilon[(p_3 - y^2)\dot{y} + p_4 x]
\end{aligned} , \tag{5}
$$

where $\alpha, \beta, \gamma, \delta$, p_1, p_2, p_3, p_4 are parameters and $\alpha, \beta, \gamma, \delta = \pm 1$. Let us illustrate recent features of program WInSet [3] to provide visualization of resonance structures. The averaged system (3) is three-dimensional and phase space of original system (5) is four-dimensional. First, let us use the following simplification. It is known that for systems with 3/2 degrees of freedom a two-dimensional Poincare map describes the behavior of original system solutions. Therefore in the first approximation we can reduce our task to a system with 3/2 degrees of freedom. According to the small parameter method, a solution of system (5) can be found in the form of power series, namely

$$y(t) = y_0(t) + \varepsilon y_1(t) + ... \tag{6}$$

where $y_0(t)$ is a solution of unperturbed equation. Substituting (6) in the first equation in (5) we obtain

$$\ddot{x} + \alpha x + \beta x^3 = \varepsilon[(p_1 - x^2)\dot{x} + p_2 y_0(t)] + O(\varepsilon^2) \qquad (7)$$

Neglecting the terms $O(\varepsilon^2)$ in (7) we get a system with 3/2 degrees of freedom which can be analyzed numerically using the Poincare map. Particularly, we can visualize structure of resonance zones.

3. NUMERICAL VISUALIZATION

We have recently updated a program WInSet [3] and added features of numerical plotting of the Poincare map for systems similar to (5) and (7) without $O(\varepsilon^2)$ terms. The program first calculates the period of oscillation of second oscillator and then using numerical computation of Cauchy problem for the entire system (5) the program performs plotting of Poincare map on plane $(x, \dot{x})$ for the obtained value of period. Dependency $y_0(t)$ for system (5) with $\gamma = 1, \delta = 1$ is expressed in terms of elliptic cosine (cn). With $\gamma = 1, \delta = -1$ this dependency is expressed in terms of elliptic sine (sn). When $\gamma = -1, \delta = 1$ then inside separatrix loops this dependency is expressed in terms of delta amplitudinis (dn). So with the new version of program WInSet we can effectively analyze perturbations given by Jacobi elliptic functions.

3.1 Poincare map

To demonstrate plotting of the Poincare map for equation like (7) consider conservative case, i.e. exclude the first term of perturbation in (7).

$$\ddot{x} + \alpha x + \beta x^3 = \varepsilon p_2 y_0(t) \qquad (8)$$

When $\alpha = \beta = \gamma = \delta = 1, \varepsilon = 0.1$ if we select initial conditions for the second oscillator as $(y = 0, \dot{y} = 12)$ then we will obtain picture of invariant curves shown on Fig. 1. The plotted image allow us make a conclusion on global behaviour of solutions of the original system. At this picture we observe two resonance zones I_{1pq}: one zone with $p = 1$ and other zone with $p = 3$.

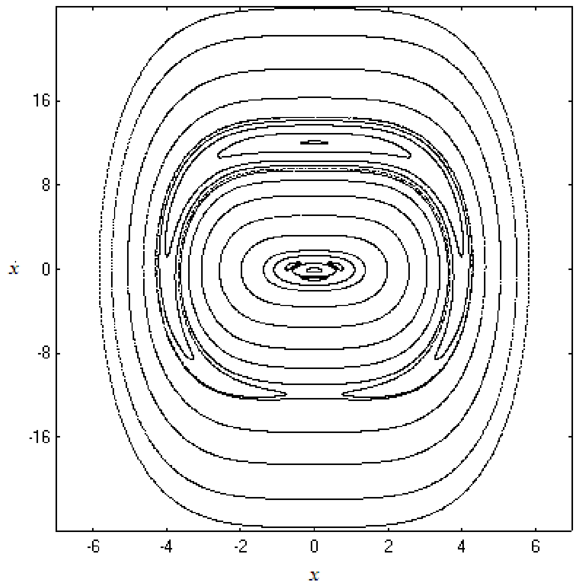

Fig. 1 Poincare map for eq. (8), $\alpha = \beta = \gamma = \delta = 1, \varepsilon = 0.1$; $y(0) = 0, \dot{y}(0) = 12$.

3.2 Visualization of averaged system

To perform detailed local analysis of solutions of equation (7) in a neighborhood of a resonance (I_{1pq}, I_{2pq}) it is necessary to explicitly calculate and investigate the averaged system (4). Below we demonstrate phase portraits of three-dimensional system (4) and their projections on (v, w) plane for the case with $\alpha = 1, \beta = 1, \gamma = 1, \delta = 1$.

Phase curves of system (4) for system (7) with $p_1 = 0.2\ p_2 = 0.2, p_3 = 0.3, p_4 = 3, \varepsilon = 0.1$ are shown on Fig. 2. In this case there is a stable equilibrium and a limit cycle in the upper half plane, (v, w).

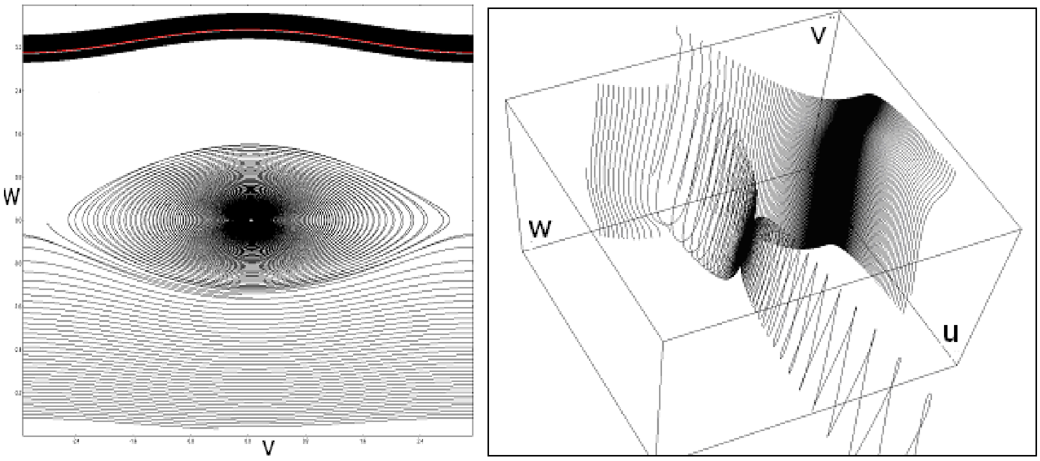

Fig. 2 Phase curves of system (4) with $\alpha = 1, \beta = 1, \gamma = 1, \delta = 1$,
$p_1 = 0.2\ p_2 = 0.2, p_3 = 0.3, p_4 = 3, \varepsilon = 0.1$

On Fig. 3 we show phase curves of system (4) with $p_1 = 0.3$ and the same values of other parameters. In this case the system has only a stable equilibrium.

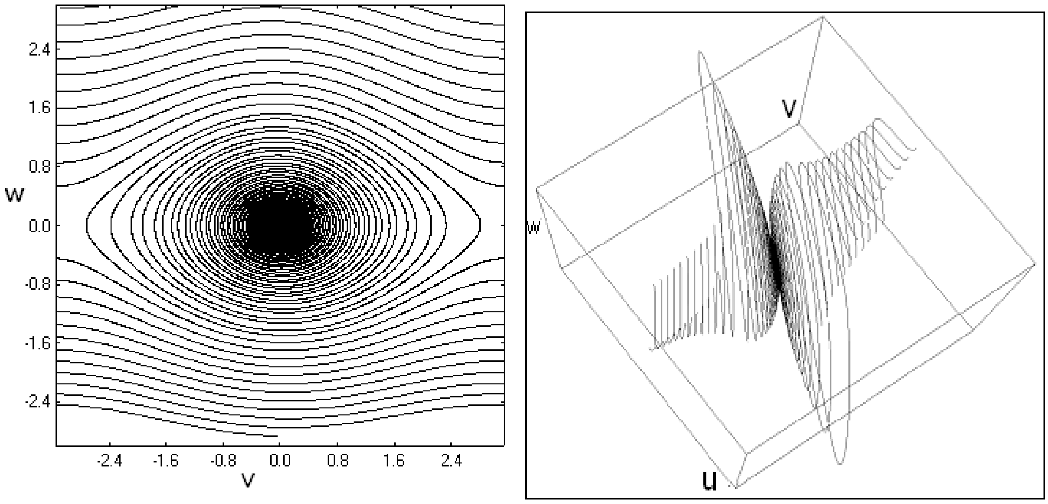

Fig. 3 Phase curves of system (4) with $\alpha = 1, \beta = 1, \gamma = 1, \delta = 1$,
$p_1 = 0.3, p_2 = 0.2, p_3 = 0.3, p_4 = 3, \varepsilon = 0.1$

If we increase the value of parameter p_1 then we get another case: a stable equilibrium and a limit cycle in the lower half plane (v, w). This case is shown on Fig. 4.

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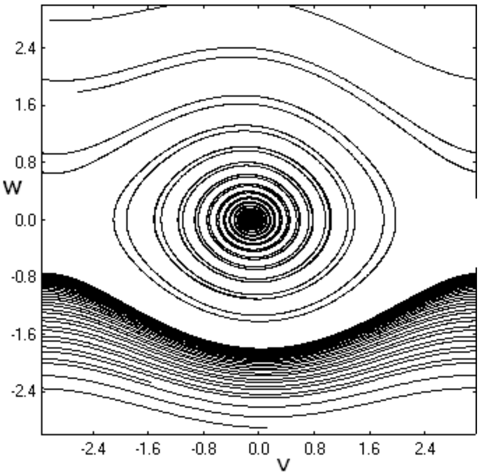

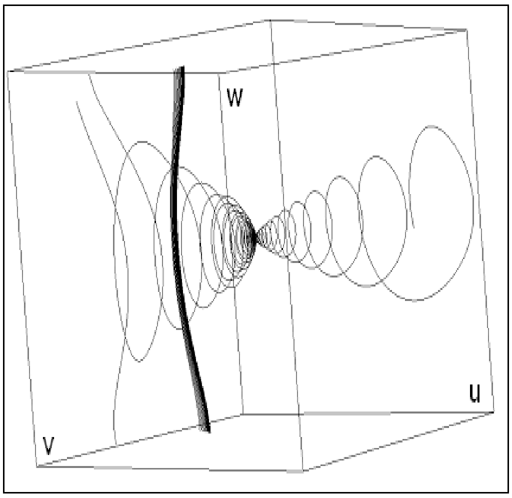

Fig. 4 Phase curves of system (4) with $\alpha = 1, \beta = 1, \gamma = 1, \delta = 1$,
$p_1 = 0.6, p_2 = 0.2, p_3 = 0.3, p_4 = 3, \varepsilon = 0.1$

REFERENCES

[1] A.D. Morozov *Resonances, Cycles and Chaos in Quasi-conservative Systems*, Regular and Chaotic Dynamics, Moscow-Izhevsk, 2005.

[2] A.D. Morozov, R.E. Kondrashov On Resonances in Systems of Two Weakly Coupled Oscillators *Int. J. Regular and Chaotic Dynamics*, Vol. 14, No 2, pp. 237-247, 2009.

[3] A.D. Morozov, T.N. Dragunov *Visualization and Analysis of Invariant Sets of Dynamical Systems*, Institute for Computer Studies, Moscow-Izhevsk, 2003.