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POLYNOMIAL VERSUS TRIGONOMETRIC EXPANSIONS FOR NONLINEAR VIBRATIONS OF SHELLS WITH DIFFERENT BOUNDARY CONDITIONS

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ABSTRACT

Geometrically nonlinear forced vibrations of circular cylindrical shells with different boundary conditions are investigated. The Sanders-Koiter nonlinear shell theory, which includes in-plane inertia, is used to calculate the elastic strain energy. The shell displacements (longitudinal, circumferential and radial) are expanded by means of a double mixed series: harmonic functions for the circumferential variable and two different formulations for the longitudinal variable; these two different formulations are: (a) Chebyshev orthogonal polynomials and (b) trigonometric functions. The same formulation is applied to study different boundary conditions; results are presented for simply supported, clamped and cantilever shells. The analysis is performed in two steps: first a liner analysis is performed to identify natural modes, which are then used in the nonlinear analysis as generalized coordinates. The Lagrangian approach is applied to obtain a system of nonlinear ordinary differential equations. Different expansions involving from 14 to 40 generalized coordinates, associated with natural modes of simply supported, clamped-clamped and cantilever shells are used to study the convergence of the solution. The nonlinear equations of motion are studied by using arclength continuation method and bifurcation analysis. Numerical responses obtained in the spectral neighborhood of the lowest natural frequency are compared with results available in literature.

INTRODUCTION

A great number of studies on geometrically nonlinear vibrations of circular cylindrical shells is available; the literature published before 2003 has been reviewed by Amabili and Païdoussis [1]. The problem is also amply discussed by Amabili in his recent monograph [2]. Here the attention is focused on large-amplitude free and forced vibrations under harmonic excitation in radial direction. In the majority of the studies Donnell's nonlinear shallow-shell theory is applied to model the problem; see, e.g. Refs. [3-6]. However, more refined classical theories have been also used, including Donnell nonlinear shell theory retaining in-plane inertia, the Sanders-Koiter (also referred as Sanders) nonlinear shell theory, the Flügge-Lur'e-Byrne nonlinear shell theory and the Novozhilov nonlinear shell theory [7-12].

The literature review shows that several methods were developed in the past for investigating nonlinear vibrations of circular cylindrical shells with different boundary conditions. Therefore, the present study is a contribution toward developing a general framework that allows studying circular shells with different boundary conditions, comparing different expansions of mode shapes.

1. STRAIN AND KINETIC ENERGY

In Fig. 1, a circular cylindrical shell having radius R, length L and thickness h is represented; a cylindrical coordinate system $(O;\ x,\ r,\ \theta)$ is considered in order to take advantage of the axial symmetry of the structure; the origin is placed at the centre of one end of the shell. Three displacement fields are shown in Fig. 1: axial $u(x,\ \theta,\ t)$, circumferential $v(x,\ \theta,\ t)$ and radial $w(x,\ \theta,\ t)$

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displacement. Geometric imperfections can be considered in the theory by means of initial radial displacements $w_0(x, \theta)$.

The nonlinear Sanders–Koiter shell theory is used, which is a classical theory derived by using the following assumptions: (i) $h<<R$ and $h<<L$; (ii) the displacements are of the order of the shell thickness h; (iii) strains are small; (iv) transverse normal stresses are negligible; (v) the normal to the undeformed middle surface remains straight and normal to the middle surface after deformation, and no thickness stretching is present (Kirchhoff–Love kinematic hypothesis); and (vi) rotary inertia is neglected.

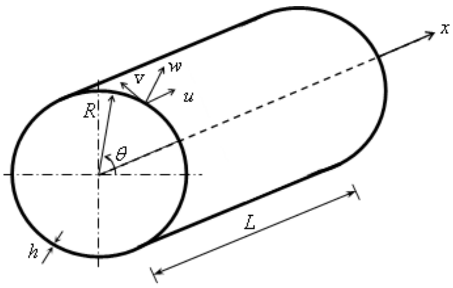


Fig. 1 Circular cylindrical shell: coordinate system and dimensions

The elastic strain energy U_S of a circular cylindrical shell is given by [2]

$$\begin{aligned} U_S = & \frac{1}{2}\frac{Eh}{1-\nu^2} LR \int_0^{2\pi}\int_0^1 \left(\varepsilon_{x,0}^2 + \varepsilon_{\theta,0}^2 + 2\nu\varepsilon_{x,0}\varepsilon_{\theta,0} + \frac{1-\nu}{2}\gamma_{x\theta,0}^2 \right) \mathrm{d}\eta\, \mathrm{d}\theta \\ & + \frac{1}{2}\frac{Eh^3}{12\left(1-\nu^2\right)} LR \int_0^{2\pi}\int_0^1 \left(k_x^2 + k_\theta^2 + 2\nu k_x k_\theta + \frac{1-\nu}{2} k_{x\theta}^2 \right) \mathrm{d}\eta\, \mathrm{d}\theta \\ & + \frac{1}{2}\frac{Eh^3}{6R\left(1-\nu^2\right)} LR \int_0^{2\pi}\int_0^1 \left(\varepsilon_{x,0} k_x + \varepsilon_{\theta,0} k_\theta + \nu\varepsilon_{x,0} k_\theta + \nu\varepsilon_{\theta,0} k_x + \frac{1-\nu}{2}\gamma_{x\theta,0} k_{x\theta} \right) \mathrm{d}\eta\, \mathrm{d}\theta + O(h^4), \end{aligned} \tag{1}$$

where $O(h^4)$ is a higher-order term in h according to the Sanders–Koiter theory.The middle surface strain-displacement relationships and changes in the curvature and torsion for a circular cylindrical shell according to Sanders-Koiter nonlinear shell theory should be found in [2, 13 and 14]. The right-hand side of equation (5) can be easily interpreted: the first term is the membrane (also referred as stretching) energy and the second one is the bending energy, while the last term couples the membrane and bending energies. E is Young's modulus and ν is the Poisson's ratio.

The kinetic energy T_S of a circular cylindrical shell, by neglecting rotary inertia, is given by

$$T_S = \frac{1}{2}\rho_S h L R \int_0^{2\pi}\int_0^1 \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) \mathrm{d}\eta\, \mathrm{d}\theta \tag{2}$$

where ρ_S is the mass density of the shell. In equation (2) the overdot denotes time derivative.

2. LINEAR VIBRATIONS. MODAL ANALYSYS

In order to carry out a linear vibration analysis, in the present section, linear Sanders–Koiter theory is considered, i.e. in equation (2), only quadratic terms are retained. The best basis for expanding displacement fields is the eigenfunction basis, but only for special boundary conditions such basis can be found analytically; generally, eigenfunctions must be evaluated numerically.

Displacement fields are expanded by means of a double series: deformation in the circumferential direction is presented by harmonic functions, Chebyshev polynomials are considered in the axial direction. Let us now consider a modal vibration, i.e. a synchronous motion:

$$u(\eta,\theta,t) = U(\eta,\theta)\, f(t),\ v(\eta,\theta,t) = V(\eta,\theta)\, f(t),\ w(\eta,\theta,t) = W(\eta,\theta)\, f(t), \tag{3}$$

where $U(\eta,\theta)$, $V(\eta,\theta)$ and $W(\eta,\theta)$ represent a modal shape. Now the modal shape is expanded in a double series in terms of Chebyshev polynomials $T_m^*(\eta)$ and harmonic functions:

$$U(\eta,\theta) = \sum_{m=0}^{M_U}\sum_{n=0}^{N} \tilde{U}_{m,n} T_m^*(\eta)\cos n\theta,\ V(\eta,\theta) = \sum_{m=0}^{M_V}\sum_{n=0}^{N} \tilde{V}_{m,n} T_m^*(\eta)\sin n\theta,\ W(\eta,\theta) = \sum_{m=0}^{M_W}\sum_{n=0}^{N} \tilde{W}_{m,n} T_m^*(\eta)\cos n\theta, \tag{4}$$

where $T_m^*(\eta) = T_m(2\eta - 1)$ and $T_m(\cdot)$ is the m-*th* order Chebyshev polynomial.

2.1 Boundary conditions

Boundary conditions are considered by applying constraints to the free coefficients of expansion (4). Some of the coefficients $\tilde{U}_{m,n}, \tilde{V}_{m,n}, \tilde{W}_{m,n}$ can be suitably chosen in order to satisfy boundary conditions.

For the simply supported shell the following boundary conditions are imposed for the mode shape:

$$w = 0, \ \ v = 0, \ \ M_x = 0, \ \ N_x = 0 \quad for \quad \eta = 0,1, \tag{5}$$

Such conditions are valid for any θ and n, therefore equations (5) are modified as follows:

$$\sum_{m=0}^{M_W} \tilde{W}_{m,n} T_m^*(\eta) = 0, \ \sum_{m=0}^{M_V} \tilde{V}_{m,n} T_m^*(\eta) = 0, \ \sum_{m=0}^{M_W} \tilde{W}_{m,n} T_{m,\eta\eta}^*(\eta) = 0, \ \sum_{m=0}^{M_U} \tilde{U}_{m,n} T_{m,\eta}^*(\eta) = 0$$
$$n = 0,1... \quad for \quad \eta = 0,1. \tag{6}$$

The linear algebraic system (6) is solved in terms of the coefficients $\tilde{U}_{1,n}, \tilde{U}_{2,n}, \tilde{V}_{0,n}, \tilde{V}_{1,n}, \tilde{W}_{0,n}, \tilde{W}_{1,n}, \tilde{W}_{2,n}, \tilde{W}_{3,n}$, $n = 0,1...$; which can be obtained exactly in terms of remaining unknown coefficients.

For the clamped-clamped shell the following boundary conditions are imposed for the mode shape:

$$w = 0, \quad w,_{\eta\eta} = 0, \quad v = 0, \quad u = 0 \tag{7}$$

For the clamped-free shell the following boundary conditions are imposed for the mode shape:

$$w = v = u = \frac{\partial w}{\partial x} = 0 \quad for \quad \eta = 0 \tag{8a}$$

$$N_x = N_{x\theta} + \frac{M_{x\theta}}{R} = M_x = Q_x + \frac{\partial M_{x\theta}}{R\partial\theta} = 0 \quad for \quad \eta = 1 \tag{8b}$$

The procedure is formally the same as for simply supported boundary conditions; however, the resulting linear systems for clamped-clamped and cantilever shells are solved in terms of the following coefficients.

2.2 Discretization

Equations (3) and (4) are inserted into the expressions of kinetic and potential energy (for the linear system); then a set of ordinary differential equations is obtained by using Lagrange equations.

An intermediate step is the reordering of variables. A vector q containing all variables is built depending on boundary conditions [12]. For simply-supported (a), clamped-clamped (b) and clamped-free (c) shell one will have:

$$q = [\tilde{U}_{0,0}, \tilde{U}_{3,0}, ..., \tilde{U}_{0,1}, \tilde{U}_{3,1}, ..., \tilde{V}_{2,0}, \tilde{V}_{3,0}, ..., \tilde{V}_{2,1}, \tilde{V}_{3,1}, ..., \tilde{W}_{4,0}, \tilde{W}_{5,0}, ..., \tilde{W}_{4,1}, \tilde{W}_{5,1}, ...] f(t) \tag{9a}$$

$$q = [\tilde{U}_{2,0}, \tilde{U}_{3,0}, ..., \tilde{U}_{2,1}, \tilde{U}_{3,1}, ..., \tilde{V}_{2,0}, \tilde{V}_{3,0}, ..., \tilde{V}_{2,1}, \tilde{V}_{3,1}, ..., \tilde{W}_{4,0}, \tilde{W}_{5,0}, ..., \tilde{W}_{4,1}, \tilde{W}_{5,1}, ...] f(t) \tag{9b}$$

$$q = [\tilde{U}_{1,0}, \tilde{U}_{2,0}, ..., \tilde{U}_{1,1}, \tilde{U}_{2,1}, ..., \tilde{V}_{1,0}, \tilde{V}_{2,0}, ..., \tilde{V}_{1,1}, \tilde{V}_{2,1}, ..., \tilde{W}_{2,0}, \tilde{W}_{3,0}, ..., \tilde{W}_{2,1}, \tilde{W}_{3,1}, ...] f(t) \tag{9c}$$

Lagrange equations for free vibrations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, ..., N_{\max} \tag{10}$$

Using (9) and considering harmonic motion, $f(t) = e^{j\omega t}$, one obtains

$$\left(-\omega^2 M + K\right) q = 0 \tag{11}$$

which is the classical nonstandard eigenvalue problem that furnishes frequencies and modes of vibration.

A modal shape corresponding to the *j*-th mode is given by equations (4), where $\tilde{U}_{m,n}, \tilde{V}_{m,n}, \tilde{W}_{m,n}$ are substituted with $\tilde{U}^{(j)}_{m,n}, \tilde{V}^{(j)}_{m,n}, \tilde{W}^{(j)}_{m,n}$, which are components of the *j*-th eigenvector of equation (11).

3. NUMERICAL RESULTS

The equations of motion have been obtained by using the *Mathematica* 6 computer software. The generic Lagrange equation *j* is divided by the modal mass associated with $\ddot{q}_j$ and then is transformed in two first-order equations. The resulting 2×*dofs* equations are studied by using the software AUTO 97 [15] for continuation and bifurcation analysis of nonlinear ordinary differential equations.

3.1 Simply supported shell

A test case of a simply supported circular cylindrical shell is analyzed. Calculations have been performed for a shell having the following dimensions and material properties: L = 0.2 m, R = 0.1 m, h = 0.247 mm, $E = 71.02 \times 10^9$ Pa, ρ = 2796 kg/m^3 and ν = 0.31, which corresponds to a case studied by several authors [5, 10, 11].

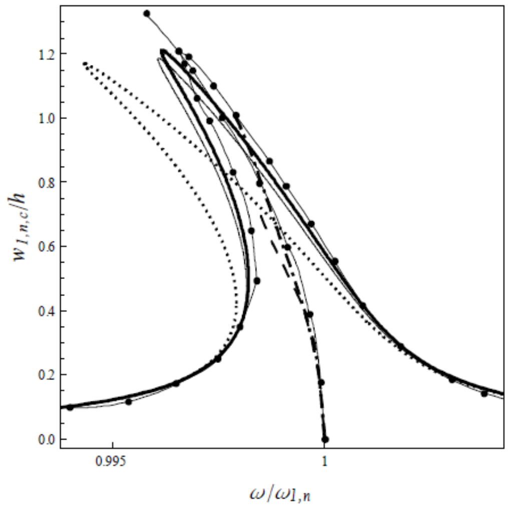


Fig. 2 Frequency response-curve for simply-supported shell. 28 dofs model (bold line) comparing with results available in literature [5, 10, 11]

Fig. 2 shows the frequency-response curve (computed by using the model with 28 dofs) of the driven mode ($A_{m=1,n=6}(t)$) with companion mode participation, namely the following modes:

$$\begin{aligned} &w\text{: } (1,n), (1,2n), (1,0)\text{-}(5,0);\\ &u\text{: } (1,n), (1,2n), (1,0)\text{-}(5,0), (3,2n);\\ &v\text{: } (1,n), (1,2n), (3,2n), (1,4n), (3,4n),(1,3n).\end{aligned} \tag{12}$$

The amplitude of the external modal excitation is $f_{1,6} = 0.0012 h^2 \rho \omega^2$ and the damping ratio is 2ζ1,6 = 0.001. The linear circular frequency of the driven and companion modes is $\omega_{1,6} = 2\pi \times 553.33$ rad/s. Fig. 2 shows reasonably good agreement between the present results and those obtained previously.

Convergence of model (12) has also been studied, but for brevity sake it is not presented in this paper. More details one should find in [16]. Frequency-response relationship with companion mode participation (i.e. the actual response of the shell) for the model (12) should also be found there.

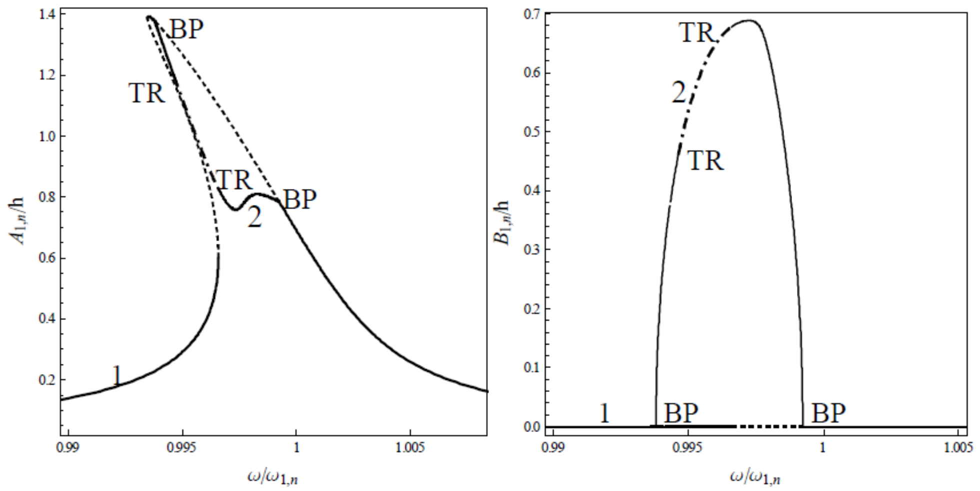
3.2 Clamped shell

Calculations have been performed for a shell having the following dimensions and material properties: L = 520 mm, R = 149.4 mm, h = 0.519 mm, $E = 1.98 \times 10^{11}$ Pa, ρ = 7800 kg/m^3 and ν = 0.3.

The response of the circular cylindrical shell subjected to harmonic point excitation of 3 N applied in the middle of the shell in the neighbourhood of the lowest (fundamental) resonance

$\omega_{1,n} = 2\pi \times 313.7$ rad/s, corresponding to mode (m = 1, n = 6), is given in Figure 3; only the principal (resonant) coordinates, corresponding to driven (a) and companion (b) modes, are shown for brevity. Calculations reported in this section have been performed by using an expansion involving 34 generalized coordinates (with companions), namely:

$$\begin{aligned} &w\text{: } (1,n), (1,2n), (3,2n), (1,0)\text{-}(9,0);\\ &u\text{: } (1,n), (1,2n), (3,2n), (1,0)\text{-}(9,0);\\ &v\text{: } (1,n), (1,2n), (3,n), (3,2n). \end{aligned} \quad (13)$$



(a) *(b)*

Fig. 3 Frequency-response curve for clamped shell with companion mode participation. —, Stable periodic solution; −·−, stable quasi-periodic solution; – –, unstable solutions; BP, pitchfork bifurcation; TR, Neimark-Sacker bifurcation.

Convergence of model (13) as well as comparison with results, available in literature, should be found in [16].

3.3 Cantilever shell

Test cases of perfect cantilever circular cylindrical shell and shell with imperfections are analyzed. Calculations have been performed for a shell having the following dimensions and material properties: L = 0.48 m, R = 0.24 m, h = 0.254 mm, $E = 4.65 \times 10^9$ Pa, ρ = 1400 kg/m^3 and ν = 0.38, which corresponds to a case studied experimentally by Chiba [17]. The mode investigated is (m=1, n=7) which has one longitudinal half-wave and 7 circumferential waves.

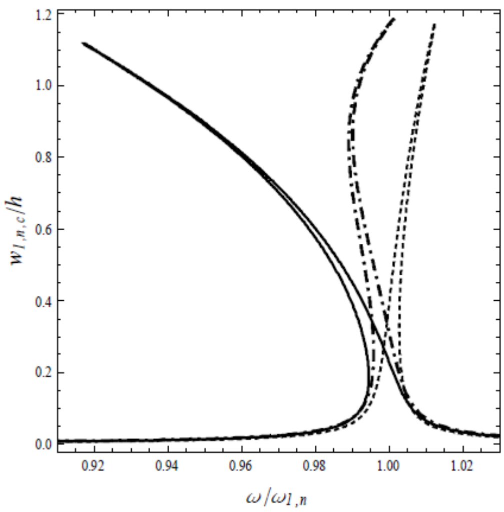


Fig. 4 Frequency-response curve for cantilever shell with imperfections having different magnitude: one thickness imperfection (dashed-dotted line), two thickness magnitude (solid line), no imperfection (dashed line).

Fig. 4 shows that presence of imperfections can significantly change the behavior of the system. Moreover, type of the system response depends also on magnitude of imperfection.

CONCLUSIONS

The response of circular cylindrical shells with different boundary conditions has been computed by using Sanders-Koiter theory. Displacement fields were expanded by means of a double series: deformation in the circumferential direction is presented by harmonic functions, Chebyshev polynomials were considered in the axial direction.

The approach used in the present study has the advantage of being suitable to be applied to different boundary conditions, of satisfying them exactly and of being very flexible to structural modifications without complication of the solution procedure. Comparison of the present study results with results available in literature was carried out and showed good agreement.

More details of the present study should be found in [16]. Detailed report on nonlinear vibrations of cantilever shells will be published soon.

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REFERENCES

[1] M. Amabili and M. P. Païdoussis, Review of studies on geometrically nonlinear vibrations and dynamics of circular cylindrical shells and panels, with and without fluid-structure interaction. *Applied Mechanics Reviews*, 56, pp. 349-381, 2003.

[2] M. Amabili, *Nonlinear Vibrations and Stability of Shells and Plates*. New York, USA: Cambridge University Press; 2008.

[3] D. A. Evensen, Nonlinear flexural vibrations of thin-walled circular cylinders. *NASA TN D-4090*; 1967.

[4] E. H. Dowell and C. S. Ventres, Modal equations for the nonlinear flexural vibrations of a cylindrical shell. *International Journal of Solids and Structures*, 4, pp. 975-991, 1968.

[5] J. C. Chen and C. D. Babcock, Nonlinear vibration of cylindrical shells. *AIAA Journal*, 13, pp. 868-876, 1975.

[6] V. D. Kubenko, P. S. Koval'chuk and T. S. Krasnopol'skaya, Effect of initial camber on natural nonlinear vibrations of cylindrical shells. *Soviet Applied Mechanics*, 18, pp. 34-39, 1982.

[7] M. Amabili, Nonlinear vibrations of circular cylindrical shells with different boundary conditions. *AIAA Journal*, 41, pp. 1119-1130, 2003.

[8] M. Amabili, Theory and experiments for large-amplitude vibrations of empty and fluid-filled circular cylindrical shells with imperfections. *Journal of Sound and Vibration*, 262, pp.921-975, 2003.

[9] P. B. Gonçalves and R. C. Batista, Non-linear vibration analysis of fluid-filled cylindrical shells. *Journal of Sound and Vibration*, 127, pp. 133-143, 1988.

[10] M. Ganapaphi and T. K. Varadan, Large amplitude vibrations of circular cylindrical shells. *Journal of Sound and Vibration*, 192, pp. 1-14, 1996.

[11] M. Amabili, Comparison of different shell theories for large-amplitude vibrations of empty and fluid-filled circular cylindrical shells with and without imperfections: Lagrangian approach. *Journal of Sound and Vibration*, 264, pp. 1091-1125, 2003.

[12] F. Pellicano, Vibrations of circular cylindrical shells: Theory and experiments. *Journal of Sound and Vibration*, 303, pp. 154-170, 2007.

[13] J. L. Sanders Jr., Nonlinear theories for thin shells. *Quarterly of Applied Mathematics*, 21, pp. 21-36, 1963.

[14] W. T. Koiter, On the nonlinear theory of thin elastic shells. I, II, III. *Proceedings Koninklijke Nederlandse Akademie van Wetenschappen*, B 69, pp. 1-54, 1966.

[15] E. J. Doedel, A. R. Champneys, T. F. Fairgrieve, Y. A. Kuznetsov, B. Sandstede and X. Wang, *AUTO 97: Continuation and Bifurcation Software for Ordinary Differential Equations (with HomCont)*. Montreal, Canada: Concordia University; 1998.

[16] Ye. Kurylov, M. Amabili, Polynomial versus trigonometric expansions for nonlinear vibrations of circular cylindrical shells with different boundary conditions. *Journal of Sound and Vibration*, 329, pp. 1435-1449, 2010.

[17] M. Chiba, Non-linear hydroelastic vibration of a cantilever cylindrical tank – I. Experiment (Empty case). *International Journal of Non-Linear Mechanics* 28, pp. 591-599, 1993.