Proceedings of the 3rd International Conference on Nonlinear Dynamics
ND-KhPI2010
September 21-24, 2010, Kharkov, Ukraine

RECONSTRUCTION OF THE THIRD ORDER DYNAMICAL SYSTEMS FROM SIGNALS

Evgeniy D. Pechuk[1]
Institute of Hydromechanics NASU
Kyiv, Ukraine

Tatyana S. Krasnopolskaya
Institute of Hydromechanics NASU
Kyiv, Ukraine

ABSTRACT

The problem of reconstruction of the deterministic dynamical system from output signals is very important. Two reconstruction methods have been used and compared. First one is the method of successive differentiation and the second is based on delay coordinates. It was firstly suggested to choose time delay parameter from the stable region of a divergence of the reconstructed system. Results show that both methods can capture regular and chaotic signals from reconstructed systems of the third order with nonlinear terms up to sixth order. Types of signals were examined with spectral methods, construction of phase portraits and Lyapunov exponents. The first method gives the solution which the power spectrum for the regular signals coincides with the output signal spectrum up to 96% for the first three peaks. The second method gives a mistake around 2 % and determines the maximum Lyapunov exponent more precisely for chaotic regimes (with a precision to $O(10^{-3})$) than the first method.

INTRODUCTION

The problem of reconstruction of deterministic dynamical system from output signals is of great importance in studying of properties of experimental signals such as acoustic signals, ECG, EEG and so on. Reconstructed dynamical system may add a significant qualitative information to chaotic data analysis. Stability conditions, bifurcation curves, all types of steady – state regimes could be studied for solutions of a reconstructed system.

Two reconstruction methods have been developed by Crutchfield and McNamara [1] and used for variety of signals later [2-4].

The first method is based on suggestion that the signal can be presented by a function that has at least three derivatives, so this is method of successive differentiation. Applying this method the dynamical system has a following form [1-4]:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \dfrac{dx_3}{dt} = F_3(x_1, x_2, x_3) \end{cases} \tag{1}$$

where $F_3(x_1, x_2, x_3)$ is a nonlinear function.

The second method of reconstruction is based on delay coordinates. We need to reconstruct the dynamical system from the time series of some state variable $x(t)$ with the fixed sampling step dt. We have series of $s_k = x(kdt)$, $k=0,1,2,\dots,N$, using value of time delay $\tau = ndt$ (which is chosen to yield optimal reconstruction [1]) we construct the dynamical system in the form [1-4]:

[1] Corresponding author. Email uzuzun@i.ua

$$\begin{cases} \dfrac{dx_1}{dt} = F_1(x_1, x_2, x_3) \\ \dfrac{dx_2}{dt} = F_2(x_1, x_2, x_3) \\ \dfrac{dx_3}{dt} = F_3(x_1, x_2, x_3) \end{cases} \tag{2}$$

where $x_1(t) = x(t)$; $x_2(t) = x(t+\tau)$; $x_3(t) = x(t+2\tau)$, $F_i(x_1, x_2, x_3)$ are nonlinear functions.

1. RECONSTRUCTION OF DYNAMICAL SYSTEMS FOR OUTPUT SIGNALS OF PENDULUM SYSTEM

Reconstruction methods are applied to the signals of a deterministic dynamical system (3) of pendulum oscillations which may have regular and chaotic regimes [5]**:**

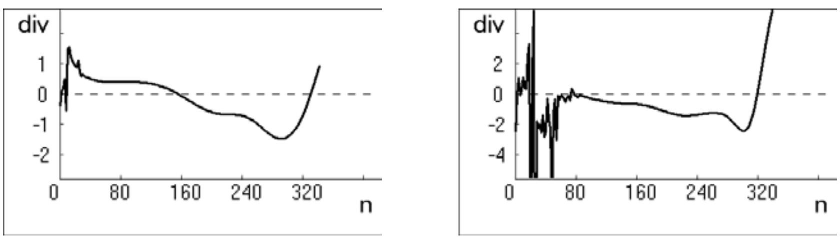
$$\begin{cases} \dfrac{dy_1}{dt} = -0.1y_1 - y_2y_3 - \dfrac{1}{8}(y_1^2y_2 + y_2^3) \\ \dfrac{dy_2}{dt} = -0.1y_2 + y_1y_3 - \dfrac{1}{8}(y_2^2y_1 + y_1^3) + 1 \\ \dfrac{dy_3}{dt} = -0.5y_2 - 0.61y_3 + F \end{cases} \tag{3}$$

Nonlinear functions $F_i(x_1, x_2, x_3)$ in the systems (1) and (2) have the following form

$$F(x_1, x_2, x_3) = a + \sum_{i=1}^{3} a_i x_i + \sum_{i,j=1}^{3} a_{ji} x_j x_i + ... + \sum_{o,m,n,k,j,i=1}^{3} a_{omnkji} x_o x_m x_n x_k x_j x_i \tag{4}$$

with nonlinear terms up to third order for the regular signals and up to the six order for the chaotic.

The traditional way to obtain time delay parameter $\tau = ndt$ for the second method of reconstruction is to use time interval when the autocorrelation function is equal to zero [2-4]. For such chosen τ the divergence of a reconstructed system may not be negative. So that we introduce other way to choose τ. Real system is nonconservative and, the divergence of systems should be negative too. For example, for the original system (3) div is equal to -0.81. In Fig. 1 the dependence of reconstructed systems divergence on n in the steady – state regimes is shown. We choose n for time delay τ from the stable region of div.

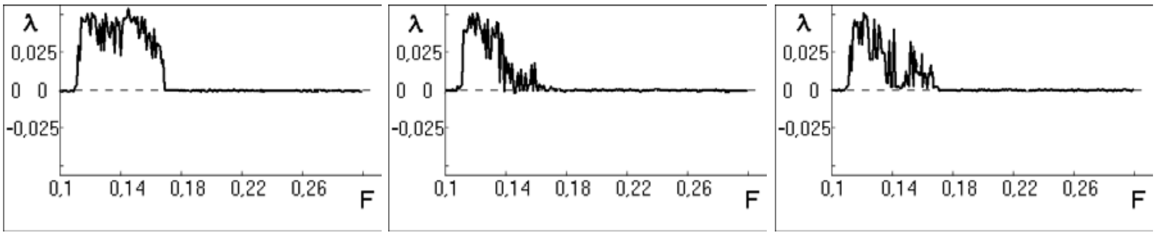


a *b*

Fig. 1 The dependence of reconstructed systems divergence on n *for regular initial signal (* $F = 0.257$ *) (a) and chaotic (* $F = 0.114$ *) (b).*

For every value of the bifurcation parameter F from the interval $0.1 \le F \le 0.3$ the reconstructed systems were built and the output signals were determined. And then the largest Lyapunov exponents [6] were calculated. For that purpose we use the fifth – order Runge – Kuttas method with the precision of $O(10^{-7})$. Initial conditions were selected in the vicinity of the original signal, and for the steady – state regime signals we choose $N = 2^{18}$, $dt = 0.004$.

The dependence of the largest Lyapunov exponent of the system (3) on values of the bifurcation parameter F is shown in Fig. 2 (a). The dependences of the largest Lyapunov exponent on F for reconstructed dynamical systems (1) and (2) are shown in Fig. 2 (b) – (c) correspondingly.



a *b* *c*

Fig. 2.The largest Lyapunov exponent of the system (3) (a), of the reconstructed systems (b) – (c).

We may see similarity of both graphs to the dependence for the original system in Fig. 2 (a) with the exception of the region $0.15 \le F \le 0.18$ where the transition to chaos occurs.

2. RECONSTRUCTED SYSTEMS FOR REGULAR OUTPUT SIGNALS

As was shown in the book [5] the solution of the pendulum system would be regular if bifurcation parameter is $F = 0.257$. We used this value and solved the system (3) in order to get the output signal. Then we reconstruct the system using the two methods. As a result the first method gives the system [7,8]

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \dfrac{dx_3}{dt} = -1.146 - 0.8x_1 - 2.211x_2 - 1.736x_3 - 0.466x_1^2 - 1.234x_1x_2 + \\ -0.507x_1x_3 - 0.119x_2^2 + 0.034x_2x_3 - 0.144x_3^2 + \\ -0.134x_1^3 - 0.08x_1^2x_2 + 0.004x_1^2x_3 - 0.041x_1x_2^2 - 0.05x_1x_2x_3 + \\ +0.006x_1x_3^2 - 0.041x_2^3 + 0.003x_2^2x_3 - 0.01x_2x_3^2 + 0x_3^3 \end{cases} \tag{5}$$

For the second method we reconstruct the system using small initial value for the delay parameter and build the dependence of the divergence on value n and choose n from the stable interval of the delay parameter (Fig. 1 (a), $n = 240$). As the result the system gets the form (6), if we take into account nonlinear terms only to the third order of nonlinearity.

Projections of the limit cycle with two loops on the plane are shown in Fig. 3 (a) – (c) for the solution of the original system (3) (Fig. 3 (a)) and the reconstructed dynamical systems (5) – (6) (Fig. 3 (b) – (c)). Since for reconstruction we use only the first variable signal phase portrait projections on the plane with the second variable only qualitatively are look like the original limit cycle with two loops. Time realizations of the first variable and their power spectrums are presented in Fig. 3 (d) – (i). Fig. 3 (d) and Fig. 3 (g) describe the solution of the original system (3), and Fig. 3 (e) – (f) and Fig. 3 (h) – (i) give the information about solutions of the dynamical systems (5) –(6).

Since power spectrum indicates the power contained at each frequency, the peak heights corresponds to the squared wave amplitudes (i.e. the wave energy) at the corresponding frequencies. The first method of reconstruction gives the solution which the power spectrum for the regular signals coincides with the output signal power spectrum up to 96% for the first three peaks. The second method gives the precision up to 98%. Also the second method determines the maximum Lyapunov exponent more precisely for chaotic regimes (with a precision to $O(10^{-3})$) than the first method.

$$
\begin{cases}
\dfrac{dx_1}{dt} = -0.266 - 2.135x_1 + 3.545x_2 - 1.574x_3 - 0.131x_1^2 + 0.685x_1x_2 + \\
- 0.352x_1x_3 - 0.393x_2^2 + 0.298x_2x_3 - 0.176x_3^2 + \\
+ 0.011x_1^3 + 0.017x_1^2x_2 - 0.042x_1^2x_3 - 0.037x_1x_2^2 + 0.093x_1x_2x_3 + \\
- 0.018x_1x_3^2 - 0.008x_2^3 - 0.035x_2^2x_3 + 0.006x_2x_3^2 - 0.007x_3^3 \\
\dfrac{dx_2}{dt} = 0.042 - 0.505x_1 - 0.427x_2 + 0.944x_3 + 0.072x_1^2 - 0.171x_1x_2 + \\
+ 0.048x_1x_3 - 0.013x_2^2 - 0.096x_2x_3 - 0.006x_3^2 + \\
- 0.005x_1^3 + 0.002x_1^2x_2 + 0.013x_1^2x_3 - 0.003x_1x_2^2 + 0.003x_1x_2x_3 + \\
- 0.015x_1x_3^2 + 0.002x_2^3 + 0.006x_2^2x_3 + 0.009x_2x_3^2 - 0.001x_3^3 \\
\dfrac{dx_3}{dt} = -0.06 + 0.31x_1 - 1.576x_2 + 1.224x_3 - 0.081x_1^2 + 0.125x_1x_2 + \\
- 0.021x_1x_3 + 0.048x_2^2 - 0.107x_2x_3 - 0.046x_3^2 + \\
+ 0.005x_1^3 - 0.009x_1^2x_2 + 0.006x_1^2x_3 - 0.006x_1x_2^2 - 0.056x_1x_2x_3 + \\
+ 0.056x_1x_3^2 + 0.008x_2^3 - 0.007x_2^2x_3 - 0.028x_2x_3^2 + 0x_3^3
\end{cases}
\qquad (6)
$$

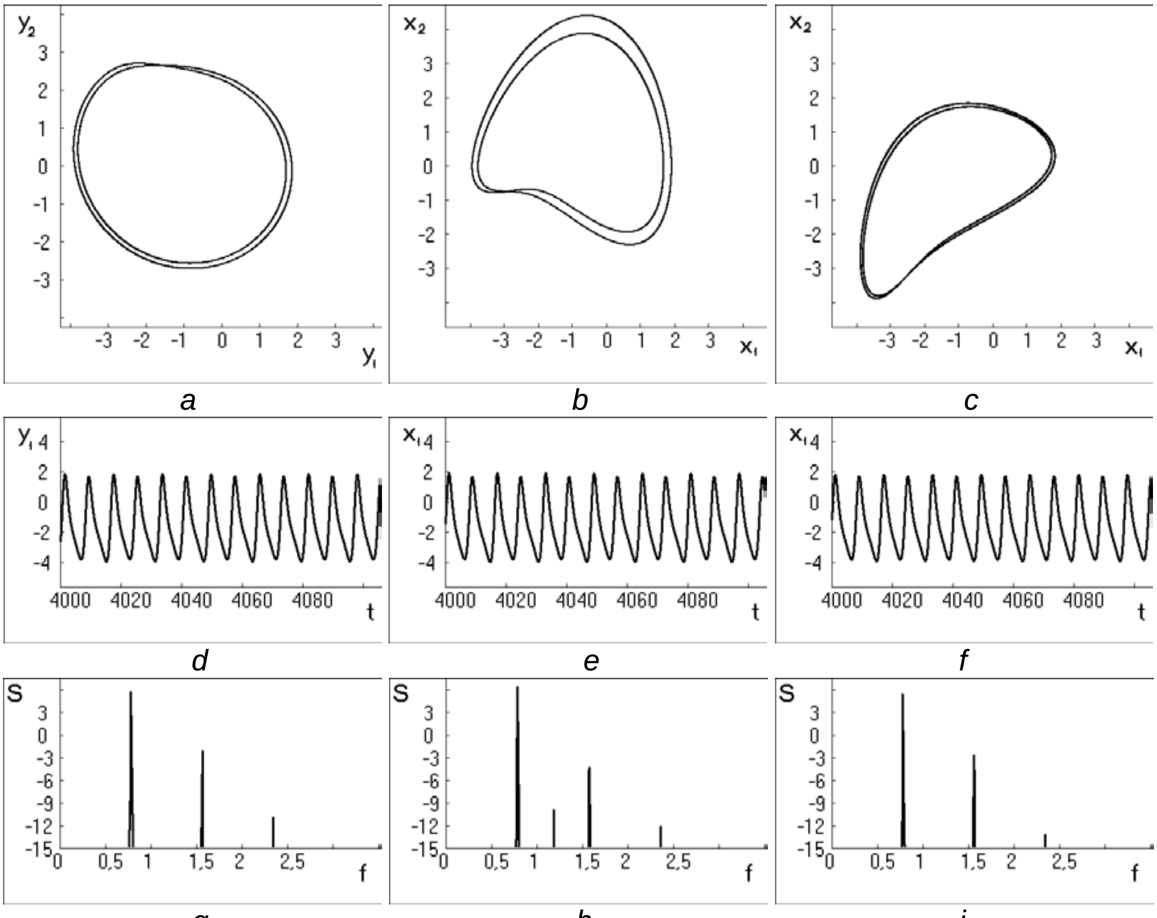


Fig. 3 The portrait of initial system (a) ($F = 0.257$), the portraits of the systems (5) – (6) (b) –(c), their time realizations (d) –(f) and power spectrums (g) –(i).

3. RECONSTRUCTED SYSTEMS FOR CHAOTIC OUTPUT SIGNALS

Now we use such parameter F for the pendulum original system when this system has the chaotic solution, namely $F = 0.114$. Then we reconstruct the system using the two methods of reconstruction with nonlinear function $F_i(x_1, x_2, x_3)$ with nonlinear terms up to the sixth order. For the second method we reconstruct the system using small initial value for the delay parameter and build the dependence of the divergence on value n and choose n from the stable interval of the delay parameter (Fig. 1 (b), $n = 120$).

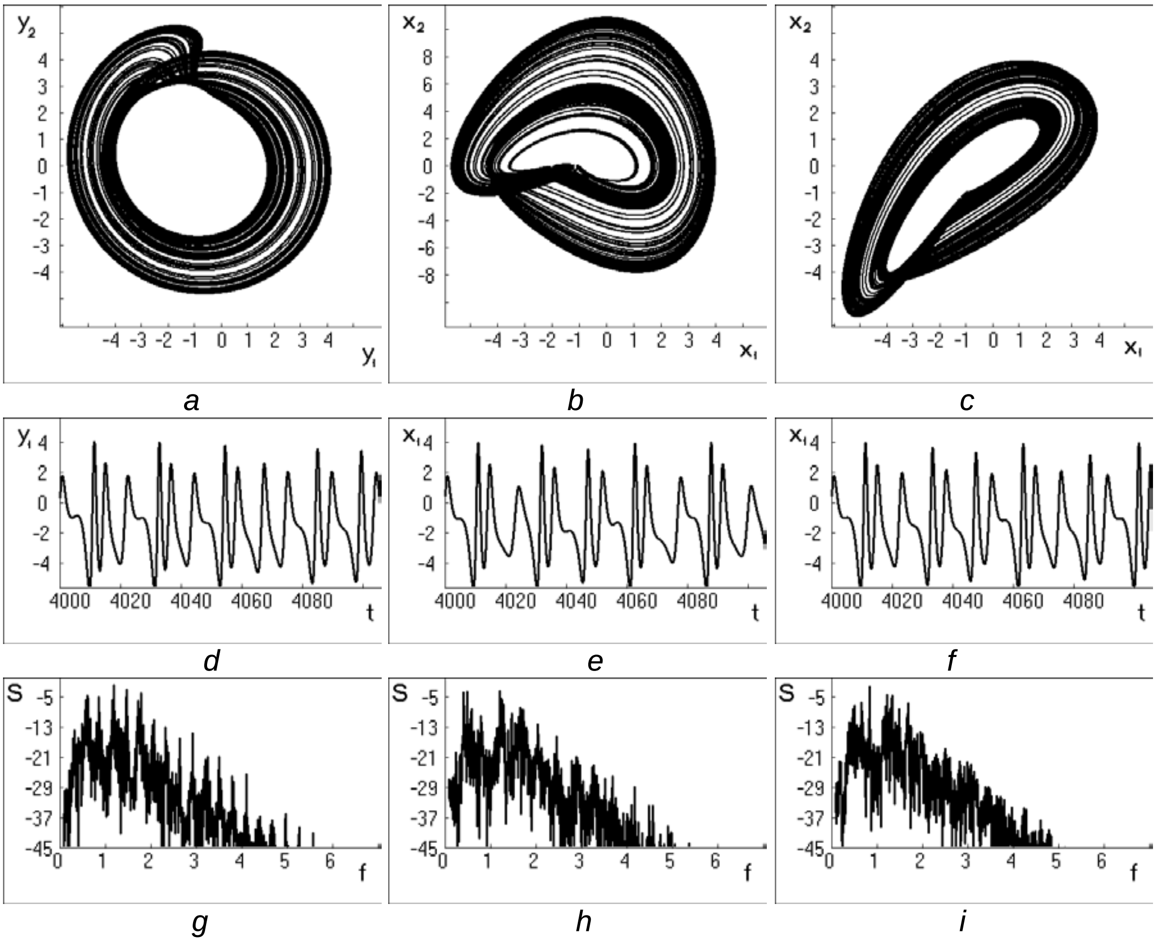
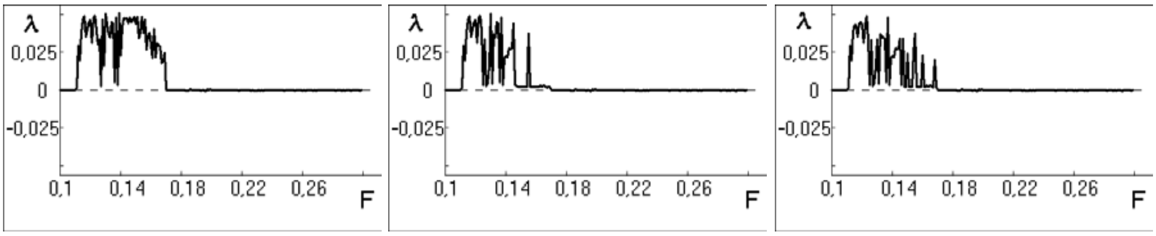


Fig. 4 The portrait of initial system (a) ($F = 0.114$), *the portraits of the reconstructed systems (b) –(c), their time realizations (d) –(f) and power spectrums (g) –(i).*

Projections of the chaotic attractor of the system (3) and of the reconstructed systems are shown in Fig. 4 (a) – (c). As could be seen from Fig. 4 the both methods qualitatively good approximate chaotic attractor of the original system (3).

Time realizations of the chaotic attractors after finished transient regimes are also similar and given in Fig. 4 (d) – (f). Power spectrums for the original signal and for the signals from the reconstructed systems are shown in Fig. 4(g) – (i) and may be approximated by the same decay function $S = -6.75 - 8.5f$.

Lyapunov exponents could be calculated directly from signals without using the dynamical systems. So that, we calculated the largest Lyapunov exponents both from original signal (Fig. 5 (a)) and from solutions of the reconstructed systems (Fig. 5 (b), (c)). Comparison those Lyapunov exponents with the given ones in Fig. 2 shows that the regions of values F , where chaotic regimes are realized, are almost the same, but Lyapunov exponents for the reconstructed signals have inside of chaotic regions more windows of regularity than the largest Lyapunov exponent for the reconstructed systems. Moreover more precisely the region of chaotic signals gives the second method of reconstruction. For example, for $F = 0.114$ the largest Lyapunov exponent for original signal is $\lambda = 0.04238$, and for signals from reconstructed systems by the first method it is equal $\lambda = 0.03368$ and by the second method is $\lambda = 0.04046$.



a *b* *c*

Fig. 5. The largest Lyapunov exponent of the initial signals (a), of the solutions of reconstructed systems (b) – (c).

CONCLUSIONS

Results show that both methods can capture regular and chaotic signals from reconstructed systems of the third order with nonlinear terms up to sixth order. Types of signals were examined with spectral methods, construction of phase portraits and Lyapunov exponents. The first method gives the solution which the power spectrum for the regular signals coincides with the output signal spectrum up to 96 % for the first three peaks. The second method gives a mistake around 2 %. And the second method determines the maximum Lyapunov exponent more precisely for chaotic regimes (with a precision to $O(10^{-3})$) than the first method.

Real systems are nonconservative and, a divergence of systems should be negative. It was suggested for the first time that the delay parameter for the second reconstruction method must be chosen from the stable region of the divergence behaviour of the reconstructed system.

The both methods qualitatively good approximate the phase portrait of chaotic attractor of the original system. Moreover, time realizations of the chaotic attractors after finished transient regimes are quiet similar. And what is more important, power spectrums for the original signal and for the signals from the reconstructed systems may be approximated by the same decay function $S = -6.75 - 8.5f$. Calculations also show that more precisely the value of bifurcation parameter for chaotic regimes gives the second method of reconstruction.

The Lypunov exponents were calculated directly from signals without using the dynamical systems. Comparison the largest Lyapunov exponent for the signals with the largest Lyapunov exponent of the systems shows that the regions of values F , where chaotic regimes are realized, are almost the same, but Lyapunov exponents from reconstructed signals have inside of the chaotic region more windows of regularity.

REFERENCES

[1] Crutchfield, J.P., McNamara, B.S. Equations of Motion from a Data Series *Complex Systems*, Vol.1, pp. 417-452, 1987.

[2] Janson, N.B., Pavlov, A.N., Kapitaniak, T., Anishenko V.S. Reconstruction of the dynamical systems from the short signals *Letters into JTP*. Vol. 25, N. 11. pp. 7-13, 1999.

[3] Anishenko, V. S. *Acquaintance with nonlinear dynamics*. Institute of Computer Science, Moscow-Izhevsk, 2002.

[4] Kouznetsov, S.P. *Dynamic chaos*. Physmatlit, Moscow, 2001.

[5] Krasnopolskaya, T.S., Shvets, A.Yu. *Regular and chaotic dynamics of the systems with limited excitation*. Institute of Computer Science, Moscow-Izhevsk, 2008.

[6] Benettin, G., Galgani, L., Strelcyn, J.M. Kolmogorov entropy and numerical experiments *Phys. Rev.*, Vol. 14, pp. 2338-2345, 1976.

[7] Kennel, M.B., Brown, R. and Abarbanel, H. D. I. Determining embedding dimension for phase-space reconstruction using a geometrical construction *Phys. Rev. A.*, Vol. 45 N. 6, p. 3403, 1992.

[8] Anishchenko, V.S., Pavlov, A.P. Global reconstruction in application to multichannel communication *Phys. Rev. E.*, Vol. 57 N. 2, pp. 2455-2458, 1998.