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RESPONSE-OPTIMAL DECELERATION OF THE ROTATION OF A SYMMETRIC FREE RIGID BODY IN A RESISTIVE MEDIUM

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ABSTRACT

The problem of time-optimal deceleration of rotation of a free rigid body is studied. It is assumed that the body contains a moving mass connected to the body by an elastic coupling with square-law friction. Low deceleration torque of viscous friction forces also acts on the rigid body. It is assumed that the body is dynamically symmetric. The optimal control law for deceleration of rotation of the rigid body in the form of synthesis, the operation time, and the phase trajectories are determined.

INTRODUCTION

Analysis of passive motion of a rigid body with a cavity filled with viscous liquid, motion of a rigid body with a moving mass connected to the body by an elastic coupling with viscous or square-law friction and motion in a resistive medium is fulfilled in [1-8]. The problem of control of rotation of “quasi-rigid” bodies via concentrated torques of forces important for application was insufficiently studied. A class of systems resulting in smooth control actions and allowing one to apply methods of singular perturbations without accumulation of “boundary-layer”-type errors was separated [2, 9-13].

The problem of time-optimal deceleration of rotation of a dynamically symmetric body connected at a point on the axes of symmetry with a mass concerning the small linear sizes by an elastic coupling with square-law friction dissipation is studied. Furthermore, low decelerating torque of a resisting medium acts on the rigid body. Rotation is controlled by the torque of forces with the bounded absolute value. The considered model continues those studied performed earlier in [2, 9-13].

1. STATEMENT OF THE PROBLEM

Based on approach [3, 13] the equations of controlled rotations in projections onto the axes of the coordinate system attached to the fixed rigid body (Euler equations) can be represented in the form [3, 5, 6,11,13]

$$A\dot{p} + (C - A)qr = M_p + FG^2 qr + Spr^6\omega_\perp - \chi Ap$$

$$A\dot{q} + (A - C)pr = M_q - FG^2 pr + Sqr^6\omega_\perp - \chi Aq \tag{1}$$

$$C\dot{r} = M_r - AC^{-1} Sr^5\omega_\perp^3 - \chi Cr$$

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Here p, q, r are the projections of the vector of absolute angular velocity $\boldsymbol{\omega}$ onto the attached axes, $\mathbf{J} = diag(A, A, C)$ is the tensor of inertia of the unperturbed body, $M_{p,q,r}$ are the projections of the vector of control torque of forces $\mathbf{M}$; and $\mathbf{G} = J\boldsymbol{\omega}$ is the angular momentum of the body; its absolute value is

$$G = |\mathbf{G}| = \left[A^2\omega_\perp^2 + C^2 r^2\right]^{1/2},\ \omega_\perp^2 = p^2 + q^2$$

For simplification of the problem the structural constraint is introduced in system (1). It is assumed that the diagonal tensor of the torque of viscous resistance forces is proportional to the tensor of moment of inertia forces; i.e., the torque of dissipation forces is proportional to the angular momentum.

$$\mathbf{M}^r = -\chi J\boldsymbol{\omega} \tag{2}$$

where χ is some constant coefficient of proportionality depending on the properties of the medium and the shape of the body. The resistance acting on the body is represented by a pair of applied forces. In this case the projections of the torque of this pair of forces on the principal axes of inertia of the body are χAp, χAq, χCr [4, 5]. Such assumption is not conflicting.

It is additionally assumed that the admissible values of the torque of control forces $\mathbf{M}$ are bounded by the sphere [13]

$$\mathbf{M}^u = b\mathbf{u},\ |\mathbf{u}| \le 1,\ b = b(t, \mathbf{G}),\ 0 < b_* \le b < b^* < \infty \tag{3}$$

where b is the scalar function bounded in the considered region of variation of the arguments t, $\mathbf{G}$, according to the conditions (3). This domain is determined a priori or can be estimated via the initial data for $\mathbf{G}$, $\mathbf{G}(t_0) = \mathbf{G}^0$.

The notations of F, S, introduced in (1), are expressed in terms of the system parameters as

$$F = m\rho^2\Omega^{-2}CA^{-3},\ S = m\rho^3\lambda\Omega^{-3}d|d|C^4A^{-4},\ d = 1 - CA^{-1} \tag{4}$$

The coefficients F, S characterize the torque of forces due to a presence of elastic element. Here m is the mass of the moving point, ρ is the radius-vector of the fixing point O_1 of the moving mass on the axis of symmetry. The constants, $\Omega^2 = c/m$, $\lambda = \mu/m = \Lambda\Omega^3$ determine oscillation frequency and velocity of their damping respectively; c is the stiffness of the elastic coupling; μ is the coefficient of square-law friction.

However, if we assume that the coupling coefficients λ and Ω are such that "free" motion of the point m resulting from the initial deviations attenuates much more rapidly than the rigid body makes one revolution, then in this case the motion of the rigid body is similar to the Euler – Poinsot motion, and the relative oscillations of the point by this motion will be small. It is supposed that

$$\Omega << \omega \tag{5}$$

In equation (5) provides introducing a small parameter into (4) and assumed stated perturbed torques to be small with purpose to apply asymptotic averaging methods. Note also that the mass m can be large, comparable with the mass of the body.

The time-optimal deceleration of rotation is formulated

$$\boldsymbol{\omega}(T) = 0,\ T \to \min_{\mathbf{u}},\ |\mathbf{u}| \le 1 \tag{6}$$

It is necessary to find the optimal control law in the form of the synthesis $u = u(t, \boldsymbol{\omega})$, the corresponding trajectory $\boldsymbol{\omega}(t, t_0, \boldsymbol{\omega}^0)$ and the operation time $T = T(t_0, \boldsymbol{\omega}^0)$, as well as the Bellman function of the problem $W = T(t, \boldsymbol{\omega}) - t$.

2. SOLUTION OF THE OPTIMAL DECELERATION PROBLEM

Note that the torque of forces due to motion of a rigid body with a moving mass connected to the body by an elastic coupling with square-law friction is internal for the fictitious body, and the torque of viscous friction forces is external.

Based on dynamic programming, the synthesis of time-optimal control has the form [13]

$$M_p = -b\frac{Ap}{G}, \; M_q = -b\frac{Aq}{G}, \; M_r = -b\frac{Cr}{G}, \;\; b = b(t, G) \tag{7}$$

Here, the following can be assumed for further simplification: $b = b(t, G)$, $0 < b_1 \le b < b_2 < \infty$.

Let us multiply the first equation of (1) by Ap, the second equation by Aq, and the third equation by Cr and sum them up. We obtain the equation of the form

$$\dot{G} = -b(t, G) - \chi G, \; G(t_0) = G^0, \; G(T, t_0, G^0) = 0, \; T = T(t_0, G^0), \; W(t, G) = T(t, G) - t$$

In the assumption that $b = b(t)$, we obtain the solution and a condition for T determination,

$$G(t) = G^0 e^{-\chi(t - t_0)} - \int_{t_0}^{t} b(\tau) e^{-\chi(t-\tau)} d\tau, \; G^0 = e^{-\chi t_0} \int_{t_0}^{T} b(\tau) e^{\chi\tau} d\tau, \; T = T(t_0, G^0) \tag{8}$$

Here, t is the current deceleration time and T is operation time.

For $b = const$ the solution to equation and boundary value problem (8) is written as

$$G(t) = \frac{1}{\chi}\left[\left(G^0 \chi + b\right)\exp(-\chi t) - b\right], \; T = \frac{1}{\chi}\ln\left(G^0 \frac{\chi}{b} + 1\right), \; t_0 = 0 \tag{9}$$

Below, case (9) is analyzed in details.

3. ANALYSIS OF AXIAL ROTATION FOR CONTROLLED BODY MOTION

Substituting known expression for G into the third equation of (1) results in a nonlinear equation with respect to r

$$\dot{r} = -r\left[bG^{-1} + A^{-2}C^{-2}Sr^4\left(G^2 - C^2r^2\right)^{3/2} + \chi\right] \tag{10}$$

Replacing the axial component of the vector of angular velocity, $r = GR$, where R is the unknown function, equation (10) is reduced to the form admitting separation of variables and trivial integration,

$$\dot{R} = -A^{-2}C^{-2}SG^4R^5\left[G^2\left(1 - C^2R^2\right)\right]^{3/2} \tag{11}$$

The vector of the angular momentum $\mathbf{G}$ upon projection onto principal central axes of inertia of the body results in the expression is $Cr = G\cos\theta$, where θ is the nutation angle. As a result, the following relation is obtained for the unknown R: $CR = \cos\theta$. Equation (11) after transition to the unknown θ can be written in the form

$$\dot{\theta} = A^{-2}C^{-6}S\sin\theta|\sin\theta|\cos^5\theta\chi^{-7}\left|\left(G^0\chi + b\right)\exp(-\chi t) - b\right|^7,\ \theta(0) = \theta^0 \tag{12}$$

The solution to this equation is written as

$$\begin{aligned}&2\sec^4\theta\cos ec\theta + 5\left(\sec^2\theta - 3\right)\cos ec\theta - 2\sec^4\theta^0\cos ec\theta^0 - \\ &-5\left(\sec^2\theta^0 - 3\right)\cos ec\theta^0 + 15\ln\left|tg\left(\pi/4 + \theta/2\right)tg^{-1}\left(\pi/4 + \theta^0/2\right)\right| = K(t)\text{'}\end{aligned} \tag{13}$$

where

$$\begin{aligned}K(t) = {}&\pm 8A^{-2}C^{-6}S\chi^{-7}\Big[-(7\chi)^{-1}\left(G^0\chi + b\right)^7(\exp(-7\chi t) - 1) + \\ &+7b(6\chi)^{-1}\left(G^0\chi + b\right)^6(\exp(-6\chi t) - 1) - 21b^2(5\chi)^{-1}\left(G^0\chi + b\right)^5(\exp(-5\chi t) - 1) + \\ &+35b^3(4\chi)^{-1}\left(G^0\chi + b\right)^4(\exp(-4\chi t) - 1) - 35b^4(3\chi)^{-1}\left(G^0\chi + b\right)^3(\exp(-3\chi t) - 1) + \\ &+21b^5(2\chi)^{-1}\left(G^0\chi + b\right)^2(\exp(-2\chi t) - 1) - 7b^6\chi^{-1}\left(G^0\chi + b\right)(\exp(-\chi t) - 1) - b^7t\end{aligned}$$

It can be assumed without losing generality that the value of θ^0 (and θ) lies in the first quarter ($0 \le \theta^0 \le \pi/2$). If θ^0 takes values in this interval, then the nutation angle also does not go beyond these limits in the course of evolution of rotation, since $\theta^* = 0$ and $\theta^* = \pi/2$ are the stationary points of equation (12).

For $A \approx C$, and θ^0 the perturbation methods can be applied in the neighborhood of stationary points; in this case these methods result in elementary expressions. For example, after the first iteration we have the following expression for θ

$$\theta(t) = \theta^0 + \frac{1}{8}\sin^2\theta^0\cos^5\theta^0 K(t) \tag{14}$$

Formula (14) provides the temporal analysis of the nutation angle for different values of the system parameters and initial data.

4. NUMERICAL ANALYSIS AND CONCLUSIONS

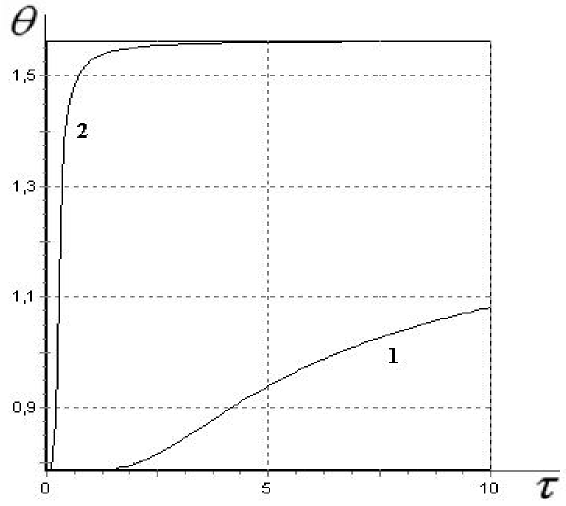
Let us consider the problem of determination of the nutation angle $\theta(t)$ in the particular case $b = const$ according to (12). Let us transform this equation to the dimensionless form. We introduce the notation

$$\tau = \chi t,\ k^* = \frac{kS^{1/7}}{A^{2/7}C^{6/7}\chi^{1/7}},\ G_0^* = \frac{G_0S^{1/7}}{A^{2/7}C^{6/7}\chi^{1/7}},\ k = b\chi^{-1} \tag{15}$$

As a result of these transformations, we obtain the equations for the nutation angle θ,

$$\frac{d\theta}{d\tau} = sign(d)\left|\left(G_0^* + k^*\right)\exp(-\tau) - k^*\right|^7\sin\theta|\sin\theta|\cos^5\theta \tag{16}$$

Equations (16) was numerically integrated for arbitrary values of G_0^*, k^* and initial angle $\theta^0 = \pi/2$ rad. The plots of variation of the nutation angle θ are shown in Figs. 1-2. Figure 1 corresponds to the dynamically prolate body, and Fig. 2 to the oblate body.



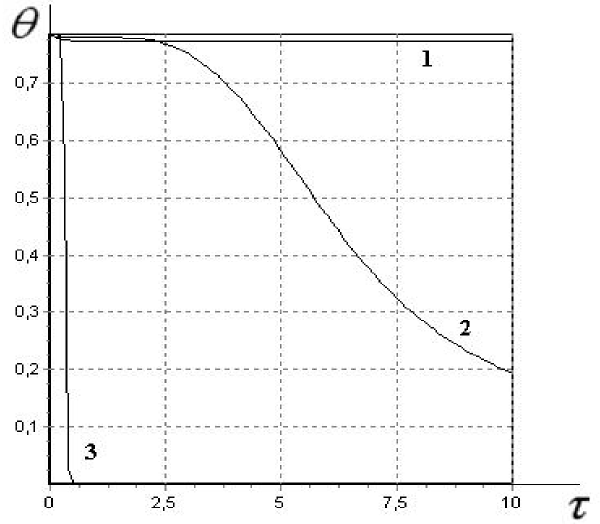


Fig. 1 *Fig.2*

Fig. 1 corresponds to the dimensionless initial value of the angular momentum $G_0^* = 1$. Curves 1, 2 and 3 were calculated for arbitrary values of $k^* = 0.1, 1, 10$ respectively. According to the calculation for dynamically prolate body ($A > C$) the nutation angle tends to a limiting value $\pi/2$ rad. Numerical interval of the dimensionless time $\tau \le 10$ is shown in Fig. 1. According to Fig. 2 it can be seen that under the essential action of the dimensionless coefficient of control torque of forces ($k^* = 10$) the nutation angle reaches the limiting value fast. In addition the body has time to brake since the operation time is the current deceleration time order over. The more smaller the value k^*, the more slowly the axis of symmetry of the body tends to the limiting position, though the body has a time to brake in the calculated time interval in all cases.

The variation of the nutation angle for dynamically oblate body was numerically studied ($A < C$). The graphs of variation of the function $\theta(t)$ for value $G_0^* = 1$ are shown in Fig. 2. Curve 1 corresponds to value $k^* = 0.1$, curve 2 corresponds to value $k^* = 1$, and curve 3 corresponds to value $k^* = 10$. According to curves 2 and 3 dynamically oblate body tends to its stable limiting position of the rotation axis corresponded to $\theta \to 0$ rad. It can be seen that the character of the tendency depends on the value of the dimensionless coefficient of the control torque of forces. The more larger this coefficient, the more faster the axis of the body tends to limiting position. In addition the operation time decreases essentially.

The numerical computation shows that the character of behavior of the function $\theta(t)$ in given problem coincides with the character of behavior of the function of the nutation angle variation for the rigid body with the moving internal masses [2].

Therefore the direction of the angular momentum vector $\mathbf{G}$ in the coordinate system fixed to the body approached a steady state along the axis corresponding to the largest moment of inertia.

CONCLUSIONS

The problem of the synthesis of time-optimal deceleration of rotation of the dynamically symmetric rigid body with a moving mass connected to the body by an elastic coupling with square-law friction in the resistive medium, is studied analytically and numerically. In the framework of the asymptotic approach, the control, the operation time (Bellman function), and the nutation angle are determined. The qualitative properties of optimal motion are established.

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