# STABILITY ANALYSIS OF PERIODIC SOLUTIONS FOR A CLASS OF FRACTIONALLY DAMPED SYSTEM OF ENGINEERING INTEREST

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### ABSTRACT

In this paper, a novel method is outlined to investigate the stability of periodic solutions for a class of dynamical systems with fractional derivative based damping. The present method essentially replaces the given fractional derivative based system by an equivalent partial differential equation , which is further approximated by a set of ordinary differential equation. The stability analysis is then carried on with the set of ordinary differential equations which is well established and well understood.

### INTRODUCTION AND BASIC THEORY

The use of passive vibration elements to mitigate excessive level of vibration in structures and machines is a well established and accepted technique. These damping elements generally are polymeric in nature, for example elastomeric rubber. Viscous damping models, though being widely used in literature and practice doesn't turn out to be accurate enough to capture structural damping behaviors of these materials. Polymers have a strong dependency of their parameters on the frequency of vibrations and it is seen that the use of fractional derivative based damping model can circumvent these modeling difficulties. This has spurred the interest of researchers for fractional derivative based models, which are now being considered as one of the most effective techniques to model materials having memory or hereditary characteristics. Earlier works on Fractional Derivative based damping were done by Bagley and Torvik [1-3] who showed that half-order fractional derivative model describe the frequency dependence of polymer based damping material very well. Koeller [4] considered a fractional calculus model to describe creep and relaxation for viscoelastic materials. Some notable contributions on the developments of analytical techniques and numerical methods on fractional order systems can be found in [5-8]. There exist different ways of defining fractional derivative. We adopt, for our problem, the Riemann-Liouville definition, as stated below

$$D_a^{\alpha}\left[x(t)\right] = \frac{d^{\alpha}x(t)}{d(t-a)^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1}$$

where *n* is the smallest integer greater than or equal to  $\alpha$  and  $\Gamma$  represents the gamma function. Structural damping is best represented when coefficient  $\alpha$  lies in the range  $0 < \alpha < 1$ . For systems starting from rest i.e. a = 0, like the one which would be discussed here, where  $x(t) \equiv 0$  for t < 0, we have  $D^{\alpha}(-\infty) = D^{\alpha}(0)$ . We will drop the a-subscript and all fractional derivative will be based on a = 0. The above expression of fractional derivative now becomes

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$$D^{\alpha}\left[x(t)\right] \equiv \frac{d^{\alpha}}{dt^{\alpha}} x(t) \equiv \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{0}^{t} \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau.$$
 (2)

In the present treatise we would choose half order derivative, i.e. when  $\alpha = \frac{1}{2}$ , as this best captures the damping characteristics of rubber like material. The expression of fractional derivative in that case reduces to

$$D^{\frac{1}{2}}[x(t)] = \frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} x(t) = \frac{1}{\Gamma(\frac{1}{2})} \frac{d}{dt} \int_{0}^{t} \frac{x(\tau)}{(t-\tau)^{\frac{1}{2}}} d\tau.$$
(3)

So the governing equation of motion of a single degree of freedom system having half order fractional derivative based damping and linear stiffness element driven by a harmonic load can be stated as

$$mD^{2}[x(t)] + c_{\rm df}D^{\frac{1}{2}}[x(t)] + kx(t) = F_{0}\cos(\omega t),$$
(4)

where  $m, c_{df}, k, F_0, \omega$  are respectively mass, damping coefficient, spring stiffness, amplitude of

impressed harmonic forcing and forcing frequency. Operator  $D^2$  denotes second derivative wrt. time t. However the price we pay in using fractional derivative based law for accurately modeling structural damping is by increasing the complexity of the systems. The governing equations of an oscillator with fractional law based damping no longer remains an ordinary differential equation (ODE). It becomes a FDE (Fractional differential equation) which actually is an integro-differential equation (IDE) or can also be labeled as a delay differential equation (with distributed delay) resulting in significant reduction in analytical tractability. Studies otherwise straightforward for a viscouslydamped oscillator become fairly non-trivial for its fractionally damped counterpart. For example stability analysis for a viscous damped oscillator having time varying coefficients can easily be obtained by invoking the celebrated Floquet theory, a detailed account of which can be obtained in [9,10]. However the stability analysis for fractionally damped system having time varying coefficients still continues to remain an area not well addressed and to the best of author's knowledge very little or almost no work has been done having an engineering flavor. However we often arrive at situations where carrying out such analysis becomes a mandatory check for qualitative treatment of such systems like while analyzing the stability behavior of periodic solutions of fractionally damped nonlinear system. In this article a method is discussed which enables us to obtain a qualitative treatment of stability behavior of a fractionally damped nonlinear oscillator. The method can also be applied for stability analysis of parametric fractional order differential equation. Before embarking on the problem of interest it seems logical to have a quick perfunctory glance at the formulation of stability analysis of periodic solutions for linear and nonlinear systems with time varying coefficients describable by ordinary differential equation.

### 1. STABILITY ANALYSIS OF STEADY STATE SOLUTION

The stability of the periodic solutions obtained by using Galerkin projection based methods can be investigated by perturbing the state variables about the steady state solutions. Consider the governing equation of a general nonlinear system, whose stability characteristics of steady state response needs to be evaluated, given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{P}(t), \tag{5}$$

where **M**, **C** and **K** are respectively mass, stiffness and damping matrices of size  $n \times n$ , **F** and **P** are vectors of size  $n \times 1$  containing nonlinear terms and impressed forcing. Let **Q**<sub>0</sub> be the steady state solution of the system, such that  $\mathbf{q}_0(t+T) = \mathbf{q}_0(t)$ . Perturbing the obtained steady state solution **Q**<sub>0</sub> by  $\Delta \mathbf{q}$  we get the incremental equation of the following form

$$\mathbf{M}\Delta\ddot{\mathbf{q}} + \mathbf{\overline{C}}\Delta\dot{\mathbf{q}} + \mathbf{\overline{K}}\Delta\mathbf{q} = \mathbf{0},\tag{6}$$

where

$$\overline{\mathbf{K}} = \mathbf{K} + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)_{(\mathbf{q}_0, \dot{\mathbf{q}}_0)}, \ \overline{\mathbf{C}} = \mathbf{C} + \left(\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{q}}}\right)_{(\mathbf{q}_0, \dot{\mathbf{q}}_0)}$$

The stability of the steady state solutions corresponds to the stability of the incremental equation (6), which is a linear ordinary differential equation with periodic coefficients in  $\overline{\mathbf{K}}$  and  $\overline{\mathbf{C}}$ . Equation (6) can be rewritten as

$$\mathbf{X} = \mathbf{Q}(t)\mathbf{X},\tag{7}$$

where

**XAq** 
$$[\Delta \dot{\mathbf{q}} ]^{\mathrm{T}}, \mathbf{Q}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\overline{\mathbf{K}}\mathbf{M}^{-1} & -\overline{\mathbf{C}}\mathbf{M}^{-1} \end{bmatrix}, \mathbf{0} \text{ and } \mathbf{I} \text{ are identity matrices of }$$

order  $n \times n$ . Since each component of  $q_0$  is a periodic solution of t with period T.

Each element of  $\mathbf{Q}$  s also a periodic function with the same period. For equation (7) there exists a fundamental set of solutions

$$\mathbf{y}_{\mathbf{k}} = [y_{1k}, y_{2k}, \dots, y_{Nk}], k = 1, 2, \dots \mathbf{N},$$
(8)

where N = 2n, n being the degrees of freedom of the system. This fundamental set can be expressed in a matrix called fundamental solution matrix **Y**,

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1N} \\ y_{11} & y_{11} & \cdots & y_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ y_{N1} & y_{N2} & \cdots & y_{NN} \end{bmatrix}.$$

It can easily be verified that Y satisfies matrix equation

$$\dot{\mathbf{Y}} = \mathbf{Q}(t)\mathbf{Y}.$$
(9)

Since  $\mathbf{Q}(t+T) = \mathbf{Q}(t)$ , therefore the following relation also holds

$$\mathbf{Y}(t+T) = \mathbf{P}\mathbf{Y}(t),\tag{10}$$

where **P** is a non-singular constant matrix called the transition matrix. Floquet theory states that stability of the periodic solution of the system given by equation (5) is governed by the eigenvalues of the matrix **P**. If all the moduli of the eigenvalues of **P** are less than unity, the motion is bounded and the solution is stable. The transition matrix **P** can be numerically evaluated by using initial condition Y(0) = I. Then we get P = Y(T), whose eigenvalues dictates the stability of the system.

### 2. PROBLEM FORMULATION

The method adopted for stability analysis in this article hinges mainly on the formulation given in [11], wherein Singh et al. used it for their numerical scheme for fractional order system. It has already been stated that fractionally damped system are in fact infinite dimensional system. The present scheme essentially replaces this original infinite dimensional system with an equivalent infinite dimensional system which in this case is a partial differential equation (PDE). We then take recourse to the Galerkin projection technique. The Galerkin projection method using suitably chosen shape functions ideally reduces a PDE to an infinite system of ordinary differential equations (ODEs). However for a practicable analysis we introduce an approximation at this stage by reducing the PDE to a finite numbers of ODEs which, is adequate to capture the essential dynamics of the original system. Once this set of ODEs is obtained it then becomes a routine exercise to carry out the stability analysis which is already illustrated in the previous section. The method can be described in details as follows. Consider the PDE in *t* with a free parameter  $\xi$ , which could also be viewed as an ODE.

$$\frac{\partial}{\partial t}u(\xi,t) + \xi^{\frac{1}{\alpha}}u(\xi,t) = \delta(t), \quad u(\xi,0^{-}) \equiv 0,$$
(11)

where  $\alpha > 0$  and  $\delta(t)$  is the Dirac delta function. The solution is

$$u(\xi,t) = h(\xi,t) = e^{\xi^{-\frac{1}{\alpha}t}},$$
(12)

where the notation  $h(\xi, t)$  is used to denote impulse response function. On integrating the function h with respect to  $\xi$  between 0 and  $\infty$  we get a function only of t, given by

$$g(t) = \int_{0}^{\infty} h(\xi, t) d\xi = \frac{\Gamma(1+\alpha)}{t^{\alpha}}.$$
(13)

Now if we replace the forcing  $\delta(t)$  in equation (11) with some sufficiently well-behaved function  $\dot{x}(t)$ , then the corresponding impulse response r(t) of the same system, again starting from rest at t = 0, is (the last two expressions are equivalent)

$$r(t) = \int_{0}^{t} g(t-\tau)\dot{x}(t)d\tau = \Gamma(1+\alpha)\int_{0}^{t} \frac{\dot{x}(t)}{(t-\tau)^{\alpha}}d\tau = \Gamma(1+\alpha)\int_{0}^{t} \frac{\dot{x}(t-\tau)}{t^{\alpha}}d\tau.$$
 (14)

On comparison with equation (2) we find that  $r(t) = \Gamma(1-\alpha)\Gamma(1+\alpha)D^{\alpha}[x(t)]$ . In this way an  $\alpha$  order derivative has been replaced by operations which involve solving the PDE in  $u(\xi, t)$ ,

$$\frac{\partial}{\partial t}u(\xi,t) + \xi^{\frac{1}{\alpha}}u(\xi,t) = \dot{x}(t), \quad u(\xi,0^{-}) \equiv 0$$
(15)

By integrating it is possible to find

$$D^{\alpha}\left[x(t)\right] = \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_{0}^{\infty} u(\xi,t)d\xi, \qquad (16)$$

which is the expression to be evaluated. The system chosen above was first prompted by Chatterjee [12] which later was used in [11]. Equation (15) represents an infinite dimensional system, and so we have replaced one infinite dimensional system (fractional derivative) with another. The advantage gained is that we can use a Galerkin projection to reduce equation (15) to a finite dimensional system

of ODEs. We, next assume  $u(t) = \sum_{i=1}^{n} a_i(t) \Phi_i(\xi)$  to be an approximate solution to the equation (15),

where *n* is finite, the shape functions  $\Phi_i(\xi)$  are user defined and the  $a_i$  are the unknowns to be solved for. We substitute approximate expression for  $u(\xi, t)$  in the equation (15) and define

$$R(\xi,t) = \sum_{i=1}^{n} \left\{ a_i(t) \Phi_i(\xi) + \xi^{\frac{1}{\alpha}} a_i(t) \Phi_i(\xi) \right\} - \dot{x}(t), \qquad (17)$$

where  $R(\xi, t)$ , is the residual. The residual is made orthogonal to the shape functions, yielding n equations

$$\int_{0}^{\infty} R(\xi, t) \Phi_{m}(\xi) d\xi = 0, \qquad m = 1, 2....n.$$
(18)

Equations (18) constitute n ODEs, which can be written in the form

$$\mathbf{A}\dot{\mathbf{a}} + \mathbf{B}\mathbf{a} = \mathbf{c}\dot{x}(t),\tag{19}$$

where **A** and **B** are  $n \times n$  matrices **a** is  $n \times 1$  vector containing  $a_i$  s and **c** is also a  $n \times 1$ . The entries of **A**, **B** and **c** can be expressed in terms of indicial notations as

$$A_{mi} = \int_{0}^{\infty} \Phi_m(\xi) \Phi_i(\xi) d\xi = 0, \quad B_{mi} = \int_{0}^{\infty} \xi^{\frac{1}{\alpha}} \Phi_m(\xi) \Phi_i(\xi) d\xi, \quad c_m = \int_{0}^{\infty} \Phi_m d\xi.$$

So given a system, linear or nonlinear with half order fractional derivative based damping shown as

$$\ddot{x} + f(x, \dot{x}, D^{\frac{1}{2}}x, t) = 0.$$
(20)

We can replace it after approximating the terms containing fractional derivative by the following expression

$$D^{\frac{1}{2}}\left[x(t)\right] = \frac{1}{\Gamma(1/2)\Gamma(3/2)} \int_{0}^{\infty} u(\xi,t) d\xi \approx \frac{1}{\Gamma(1/2)\Gamma(3/2)} \mathbf{c}^{\mathrm{T}} \mathbf{a}, \qquad (21)$$

and then augmenting the equation with equation (19). This is shown in equation (22),

$$\begin{cases} \ddot{x} + f_1(x, \dot{x}, \mathbf{c}, \mathbf{a}, t) = 0\\ \mathbf{A}\dot{\mathbf{a}} + \mathbf{B}\mathbf{a} = \mathbf{c}\dot{x}(t), \end{cases}$$
(22)

The size of vectors  $\mathbf{a}$  and  $\mathbf{c}$  depend on the numbers of shape functions chosen to approximate the expression containing fractional derivative. With a little effort equation (22) can be expressed in the state space form as

$$\dot{\overline{x}} + \overline{f_1}(\overline{x}, t) = 0, \tag{23}$$

where  $\overline{\mathbf{x}} = [\mathbf{x}, \mathbf{a}]^{\mathrm{T}}$  is the vector of extended state variable and  $\overline{f_1} = [f_1, \mathbf{A}^{-1}\mathbf{c}\dot{\mathbf{x}}(t) - \mathbf{A}^{-1}\mathbf{B}\mathbf{a}]^{\mathrm{T}}$  is the augmented function vector. With the above formulation at hand it now becomes a routine exercise to carry out the stability analysis in a manner as laid out in section 1.

### 2.1 Example.

Consider the equation as shown below

$$m\ddot{x} + c_{\rm df} D^{\frac{1}{2}} x + (\delta + \varepsilon \cos(t)) x = 0, \qquad (24)$$

This can be treated as a fractionally damped non-autonomous equation or it could be the incremental equation obtained after linearizing some equation about its steady state solution in the same way as we arrive at equation (6). It should be noted that for the second case it suffices to have the incremental equation as this only dictates the stability behavior. So the parent equation is not mentioned here. We first arrive at the state space form as shown in equation (23) by choosing 12 shape functions to approximate the fractional order term, wherein **A** and **B** matrices are of size  $12 \times 12$  and **c** is a  $12 \times 1$  vector. Due to space constraint **A**, **B** and **c** are not reproduced here. The following case studies are done to establish the proposed technique.

#### 2.1.1 Case (a) : Stable

The following values for the parameters of the system are considered for the case study.  $m = 1, c_{df} = 0.4, \delta = 0.25, \varepsilon = 0.5$ . The eigenvalues of the transition matrix (refer section 1) are obtained as  $\mu = [-0.42, -0.24, 0.57, 0.04, 0.92, 0.98, 0.99, 0.99, 0, 0, 0, 0, 0]^T$ . The analysis based on the present method states the system is stable as modulli of all the eigenvalues are less than unity. A numerical solution of the systems response confirms this result (refer Fig. 1)



Fig. 1 The displacement time plot showing bounded system response

### 2.1.2 Case (b) : Unstable

For the second example we choose  $m = 1, c_{df} = 0.4, \delta = 0.25, \varepsilon = 1$ . The eigenvalues of the transition matrices (having size  $12 \times 12$ ) is shown below:

 $\mu = [-1.917, 0.54, 0.92, 0.98, 0.999, 0.998, -0.0643, 0.0 \oplus 1, 0, 0, 0, 0, 0]^{T}$ . Our analysis predicts that the steady state solution is unstable as the modulus of first eigenvalue is greater than unity. We confirm this result by numerically obtaining the displacement vs. time plot which depicts its unbounded nature. (refer the Fig. 2)



Fig. 2 The displacement time plot showing unbounded system response

### CONCLUSIONS

A method for obtaining stability of fractionally damped system with periodic coefficients is obtained. The benchmark problems carried out establishes the method to be effective and easy to implement.

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