Conference Proceedings

the 10th International Conference on

Mathematical Methods in Electromagnetic Theory

Dniepropetrovsk, Ukraine

September 14-17, 2004

FRACTIONAL CURL OPERATOR IN RADIATION PROBLEMS

M.V. Ivakhnychenko, E.I. Veliev

IRE NASU, ul. Proskury 12, Kharkov 61085, Ukraine
E-mail: mxcd@mail.ru

Abstract – **The fractional curl operator and its possible applications in electromagnetic problems are discussed. Specifically, we consider radiation from sheet and line current distributions. Applying fractionalized curl operator we obtain new "fractional" fields and corresponding "fractional" currents, and analyze their physical meanings. "Fractional" field can be considered as "intermediate" one between the original and dual fields. It can be treated as an extension of conventional duality principle. Expressions for the "fractional" fields and relation between "fractional" currents and the original currents are presented.**

I. FRACTIONAL CURL OPERATOR

Fractional curl operator $curl^\alpha$ was introduced in [1]. Parameter α can be, in general, a complex number. In this paper we will use the presentation for the fractional curl for the functions represented by exponents [3]. This presentation will be used in two-dimensional radiation problems.
The technique for obtaining fractionalized operator was described in [1]. Following this recipe we can get the presentation for $curl^\alpha \vec{E}$ where the vector function $\vec{E}$ is expressed via exponents as $\vec{E} = \vec{z}e^{ik(x\cos\phi + y\sin\phi)}$:

$$curl^\alpha[e^{ik(x\cos\phi+y\sin\phi)}] = (ik)^\alpha[\sin\frac{\pi\alpha}{2}\sin\phi\vec{x} - \sin\frac{\pi\alpha}{2}\cos\phi\vec{y} + \cos\frac{\pi\alpha}{2}\vec{z}]e^{ik(x\cos\phi+y\sin\phi)} \quad (1)$$

Consider the source-free Maxwell's equations

$$(ik_0)^{-1}curl(\eta_0\vec{H}) = -\vec{E}; \quad (ik_0)^{-1}curl\vec{E} = \eta_0\vec{H} \quad (2)$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the intrinsic impedance of the medium.

Using fractional curl operator we can derive a new set of solutions of Maxwell's equations in the following form [1]:

$$\vec{E}^\alpha = (ik_0)^{-\alpha}curl^\alpha\vec{E}; \quad \eta_0\vec{H}^\alpha = (ik_0)^{-\alpha}curl^\alpha(\eta_0\vec{H}) \quad (3)$$

If $\alpha = 0$ or $\alpha = 1$ in the above expression, we get the original $\vec{E}^\alpha|_{\alpha=0} = \vec{E}$, $\eta_0\vec{H}^\alpha|_{\alpha=0} = \eta_0\vec{H}$ or the dual fields $\vec{E}^\alpha|_{\alpha=1} = \eta_0\vec{H}$, $\eta_0\vec{H}^\alpha|_{\alpha=1} = -\vec{E}$. Therefore, fields (3) can be considered as "intermediate" of "fractional" solutions between the original fields and the dual fields.

II. RADIATION FIELDS FROM SHEET CURRENTS

Consider fields radiated from an infinitely thin layer ("sheet") of electric current, which is a surface current distributed on infinite plane $x-z$ with the volume density $\vec{j}_e$ is given by

$$\vec{j}_e = \vec{x}J_e e^{-i\psi_0}e^{i\beta_0 x}\delta(y) \quad (4)$$

where J_e is the amplitude and ψ_0 is the initial phase of the current, J_e and ψ_0 do not depend on coordinates x,z; β_0 is the propagation coefficient. It is known that electromagnetic fields can be expressed through electric and magnetic Hertz vectors $\vec{A}^e$ and $\vec{A}^m$ as follows [2]:

$$\vec{E} = -i\omega\mu\vec{A}^e + \frac{1}{i\omega\varepsilon}graddiv\vec{A}^e - rot\vec{A}^m, \quad \vec{H} = -i\omega\varepsilon\vec{A}^m + \frac{1}{i\omega\mu}graddiv\vec{A}^m + rot\vec{A}^e \quad (5)$$

In the case when no magnetic current exists these expressions yield

$$\vec{H} = rot\vec{A}^e, \quad \vec{E} = -i\omega\mu\vec{A}^e + \frac{1}{i\omega\varepsilon}graddiv\vec{A}^e \quad (6)$$

Electric Hertz vector can be expressed through Green's function as $\vec{A}^e = \int_V \vec{j}_e(r')G(r,r')dV'$, where integration is performed over the volume where the current exists (plane $x-z$); and $G(r,r') = \frac{-ie^{ik|r-r'|}}{4\pi|r-r'|}$ is the Green's function of free space. One can derive the following presentation for electric Hertz vector $\vec{A}^e$:

$$\vec{A}^e(A_x^e,0,0) = \vec{x}\frac{1}{2}J_e\frac{e^{i\beta_0 x}e^{\pm i\gamma y}}{i\gamma} = \vec{x}\frac{1}{2}J_e\frac{e^{ik(x\cos\phi \pm y\sin\phi)}}{ik\sin\phi} \tag{7}$$

where the upper sign is chosen for $y<0$ and the down is for $y>0$. We denote $\gamma = \sqrt{k^2-\beta_0^2}$ and $\beta_0 = k\cos\phi$, $\gamma = k\sin\phi$. From (6) and (7) we obtain the fields:

$$\vec{H}(0,0,H_z) = \mp\frac{1}{2}J_e e^{i\beta_0 x}e^{\pm i\gamma y}\vec{z}, \quad \vec{E}(E_x,E_y,0) = \frac{1}{2}\frac{\eta}{k}J_e e^{i\beta_0 x}e^{\pm i\gamma y}(-\gamma\vec{x}\mp\beta_0\vec{y}) \tag{8}$$

We use index "1" for these fields. That is $\vec{E}^1,\vec{H}^1$ are the fields radiated by the current given by (4). Consider a sheet of magnetic current $\vec{j}_m$ radiating the field $\vec{E}^2,\vec{H}^2$. In order to obtain the latter fields we use duality principle that is if the magnetic current is given by

$$\vec{j}_m(j_x^m,0,0) = -\eta\vec{x}J_m e^{-i\psi_0}e^{i\beta_0 x}\delta(y) \tag{9}$$

then the fields $\vec{E}^2,\eta\vec{H}^2$ are expressed through original fields $\vec{E}^1,\eta\vec{H}^1$ as follows:

$$\vec{E}^2 = \eta\vec{H}^1, \quad \eta\vec{H}^2 = -\vec{E}^1 \tag{10}$$

So we get

$$\vec{E}^2(0,0,E_z^2) = \mp\frac{1}{2}\eta J_m e^{i\beta_0 x}e^{\pm i\gamma y}\vec{z}, \quad \eta\vec{H}^2(H_x^2,H_y^2,0) = \frac{1}{2}\frac{\eta}{k}J_m e^{i\beta_0 x}e^{\pm i\gamma y}(\gamma\vec{x}\pm\beta_0\vec{y}) \tag{11}$$

Next consider the fields radiated from a composition of two sheets: electric and magnetic.

$$\vec{E} = \vec{E}^1 + \vec{E}^2, \quad \eta\vec{H} = \eta\vec{H}^1 + \eta\vec{H}^2 \tag{12}$$

Now we apply fractional curl operator to the fields $\vec{E},\eta\vec{H}$ and obtain new "fractional" fields $\vec{E}^\alpha,\eta\vec{H}^\alpha$

$$\vec{E}^\alpha = (ik_0)^{-\alpha}curl^\alpha\vec{E}; \quad \eta_0\vec{H}^\alpha = (ik_0)^{-\alpha}curl^\alpha(\eta_0\vec{H}) \tag{13}$$

Using presentation (1) for fractional curl operator we get the expressions for fractional fields $\vec{E}^\alpha,\eta\vec{H}^\alpha$:

$$\vec{E}^\alpha(E_x^\alpha,E_y^\alpha,E_z^\alpha) = \frac{1}{2}\eta e^{i\beta_0 x}e^{\pm i\gamma y}[(\frac{\gamma}{k}J_e B - J_m A\sin\phi)\vec{x} + (\mp\frac{\beta_0}{k}J_e B \pm J_m A\cos\phi)\vec{y} + (\mp J_e A \mp J_m B)\vec{z}] \tag{14}$$

$$\eta\vec{H}^\alpha(H_x^\alpha,H_y^\alpha,H_z^\alpha) = \frac{1}{2}\eta e^{i\beta_0 x}e^{\pm i\gamma y}[(-J_e A\sin\phi - \frac{\gamma}{k}J_m B)\vec{x} + (\pm J_e A\cos\phi \pm\frac{\beta_0}{k}J_m B)\vec{y} + (\mp J_e B \pm J_m A)\vec{z}] \tag{15}$$

where $A=\sin(\pi\alpha/2)$, $B=\cos(\pi\alpha/2)$. The fields $\vec{E}^\alpha,\eta\vec{H}^\alpha$ represent radiation fields from two sheets with electric and magnetic current densities, $\vec{j}_e^\alpha$ and $\vec{j}_m^\alpha$, which can be defined via discontinuities in tangential field components:

$$H_z^\alpha|_{y\to+0} - H_z^\alpha|_{y\to-0} = (j_e^\alpha)_x, \qquad E_z^\alpha|_{y\to+0} - E_z^\alpha|_{y\to-0} = -(j_m^\alpha)_x \tag{16}$$

It can be shown that

$$\vec{j}_e^\alpha = \vec{x}J_e^\alpha e^{-i\psi_0}e^{i\beta_0 x}\delta(y), \quad \vec{j}_m^\alpha = -\vec{x}\eta J_m^\alpha e^{-i\psi_0}e^{i\beta_0 x}\delta(y) \tag{17}$$

where

$$J_e^\alpha = J_e\cos\frac{\pi\alpha}{2} - J_m\sin\frac{\pi\alpha}{2}, \quad J_m^\alpha = J_e\sin\frac{\pi\alpha}{2} + J_m\cos\frac{\pi\alpha}{2} \tag{18}$$

From (15), (16) and (19) we derive the following particular cases:
(i) if $\alpha=0$ we get the original fields and $J_e^\alpha|_{\alpha=0} = J_e$, $J_m^\alpha|_{\alpha=0} = J_m$; (ii) if $\alpha=1$ the fields (15), (16) are the dual fields radiated from electric and magnetic sheets with amplitudes $J_e^\alpha|_{\alpha=1} = -J_m$, $J_m^\alpha|_{\alpha=1} = J_e$; (iii) if $0<\alpha<1$ we have "intermediate" fields due to electric and magnetic sheet "fractional" current distributions, which are related to the original currents (19). These currents can be treated as "intermediate" or "fractional" source currents.

III. RADIATION FIELDS FROM LINE CURRENTS

In the same manner we can apply fractional curl operator to radiation fields due to the line currents distributions. Let $\vec{E},\vec{H}$ be the fields radiated by two line currents $\vec{j}_e$, $\vec{j}_m$ given by

$$\vec{j}_e = \vec{z}J_e\delta(x)\delta(y), \quad \vec{j}_m = -\vec{z}\eta J_m\delta(x)\delta(y) \tag{19}$$

The fields can be defined via electric and magnetic Hertz vectors (5):

$$\vec{A}^e(0,0,A_z^e) = -\frac{1}{4i}J_e H_0^{(1)}(k\sqrt{x^2+y^2})\vec{z}, \quad \vec{A}^m(0,0,A_z^m) = \frac{\eta}{4i}J_m H_0^{(1)}(k\sqrt{x^2+y^2})\vec{z} \tag{20}$$

where $H_0^{(1)}(k\sqrt{x^2+y^2})$ is the Hankel function of the first kind.

After applying fractional curl operator we obtain new fields $\vec{E}^\alpha,\eta\vec{H}^\alpha$ corresponded to new current distributions $\vec{j}_e^\alpha$ and $\vec{j}_m^\alpha$. Using integral presentation for the Hankel function $H_0^{(1)}(k\sqrt{x^2+y^2}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Qd\beta}{\sqrt{1-\beta^2}}$ where $Q = Q(x,y,\beta) = e^{ik(\beta x+\sqrt{1-\beta^2}|y|)}$, we can obtain the presentation for fractional fields $\vec{E}^\alpha(E_x^\alpha,E_y^\alpha,E_z^\alpha)$, $\eta\vec{H}^\alpha(H_x^\alpha,H_y^\alpha,H_z^\alpha)$:

$$E_x^\alpha = \frac{k\eta}{4\pi}(-J_eA - J_mB)\int Qd\beta, \quad E_y^\alpha = \frac{k\eta}{4\pi}(J_eA + J_mB)\int\frac{Q\beta}{\sqrt{1-\beta^2}}d\beta, \quad E_z^\alpha = \frac{k\eta}{4\pi}(-J_eB + J_mA)\int\frac{Q}{\sqrt{1-\beta^2}}d\beta \tag{21}$$

$$\eta H_x^\alpha = \frac{k\eta}{4\pi}(-J_eB + J_mA)\int Qd\beta, \quad \eta H_y^\alpha = \frac{k\eta}{4\pi}(J_eB - J_mA)\int\frac{Q\beta}{\sqrt{1-\beta^2}}d\beta, \quad \eta H_z^\alpha = \frac{k\eta}{4\pi}(J_eA + J_mB)\int\frac{Q}{\sqrt{1-\beta^2}}d\beta \tag{22}$$

And amplitudes J_e^α, J_m^α of current distributions are expressed as

$$J_e^\alpha = J_e\cos\frac{\pi\alpha}{2} - J_m\sin\frac{\pi\alpha}{2}, \quad J_m^\alpha = J_e\sin\frac{\pi\alpha}{2} + J_m\cos\frac{\pi\alpha}{2} \tag{23}$$

For the limit cases $\alpha = 0$ and $\alpha = 1$ the fields $\vec{E}^\alpha,\eta\vec{H}^\alpha$ are the original and the dual fields, respectively. That is: (i) If $\alpha = 0$ we get the original currents $J_e^\alpha|_{\alpha=0} = J_e$, $J_m^\alpha|_{\alpha=0} = J_m$; (ii) if $\alpha = 1$ we obtain the dual fields due to the currents $J_e^\alpha|_{\alpha=1} = -J_m$, $J_m^\alpha|_{\alpha=1} = J_e$; (iii) if $0 < \alpha < 1$ we have "intermediate" fields due to "fractional" electric and magnetic line current distributions, which are the combination of the original currents (24).

From (23) we see that if α is chosen that $\tan(\pi\alpha_0/2) = J_e/J_m$ then fractional electric current J_e^α becomes zero. It means that for $\alpha = \alpha_0$ the fractional fields $(\vec{E}^\alpha,\eta\vec{H}^\alpha)|_{\alpha=\alpha_0}$ represent the fields radiated only from magnetic line current J_m^α.

To show another illustrative example we choose original magnetic current to be zero $J_m = 0$. In this case $J_e^\alpha = J_e\cos(\pi\alpha/2)$, $J_m^\alpha = J_e\sin(\pi\alpha/2)$. For limit values $\alpha = 0$ and $\alpha = 1$ the fractional fields $\vec{E}^\alpha,\eta\vec{H}^\alpha$ represent the fields radiated from electric or magnetic current, respectively.

IV. CONCLUSION

We have analyzed radiation from elementary fractional sources such as sheet and line fractional-current distributions. The representations for the "fractional" fields were determined with the fractional curl operator $curl^\alpha$ $(0 \le \alpha \le 1)$ applied to the fields of certain currents. In the limit cases of $\alpha = 0$ and $\alpha = 1$ the "fractional" fields yield the original and the dual fields. The expressions for new "intermediate" ("fractional") currents were derived from the "fractional" fields.

REFERENCES

[1] N. Engheta, Fractional curl operator in electromagnetics, *Microwave Optical Techn. Lett.*, 17(2), pp. 86-91, 1998.
[2] A. Ishimaru, Electromagnetic Wave Propagation, Radiation, and Scattering, Prentice-Hall, Englewood, 1991.
[3] E. Veliev, N.Engheta, Fractional *curl* operator in reflection problems, *Proc. of MMET*04*, Dniepropetrovsk, 2004.

0-7803-8441-5/04/$20.00 © 2004 IEEE