Journal of Electromagnetic Waves and Applications, Vol. 10, No. 4, 493–507, 1996

Bessel functions series in two dimensional diffraction problems

E. I. Veliev [1] and A. Okşaşoğlu

Department of Electrical and Electronics Engineering
Çukurova University
01330 Adana, Turkey

Abstract–Alternative representations as series of more elementary functions or an analytical form for the Schlömilch type series with the Bessel functions are derived. Various special cases of such representations, which arise in different diffraction problems are presented. By detailed examinations of the convergence in numerical computation, a great reduction in computation time is established. A new representation for the Bessel function is also given.

I. INTRODUCTION

Alternative representations as series of more elementary functions or an analytical form are obtained for the Schlömilch type series

$$S_k^\lambda = \sum_{n=1}^{\infty} \frac{1}{n^\lambda} J_{k+\lambda}(n\alpha), \quad \lambda \geq 0 \tag{1}$$

$$V_{k+\lambda}^{p+\mu}(\nu) = \sum_{n=1}^{\infty} \frac{1}{n^\nu} J_{k+\lambda}(n\alpha) J_{p+\mu}(n\alpha), \quad \lambda + \mu \geq 0 \tag{2}$$

where $J_{k+\lambda}(x)$ is the Bessel function, and $\alpha \in [0, \pi]$, λ, μ, ν are the real parameters. These series are of interest in a variety of diffraction problems. In particular, the special case of (2) with $\mu = \lambda$, $\nu = 2\lambda + 1$ and $\lambda = 0$, which was considered by Veliev at el. [1–3], arises in the analysis of the scattering of a plane wave on a cylindrical screen structure. For such problems, we have $\alpha = \theta$, where θ is the geometrical size of the screen. The different types of series (1) and (2) arise for related waveguide problems [4], for radiation from the flanged parallel plate waveguide [5]. It should be noted that only special cases of (1) and (2) with $k = p = 0$, $\nu = \lambda + \mu$ are given in the reference book [6].

In general, the representations (1) and (2) converge too slowly for numerical computations. The purpose of this paper is to derive an alternative representation in explicit form or in terms of more elementary functions. In addition, a new relation is established for the Bessel function as

$$\frac{J_{k+\lambda}(m\alpha)}{m^\lambda} = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^\lambda} J_{k+\lambda}(n\alpha) \frac{\sin \alpha(m-n)}{m-n}, \quad \lambda \geq 0 \tag{3}$$

[1] E. Veliev is currently a visiting professor from the Academy of sciences, Kharkov, Ukraine.

II. MATHEMATICAL DERIVATIONS IN THE ANALYTICAL FORM

In this section, we will try to give alternative representations in an analytical form for the expressions given in (1), (2), and (3).

2.1 Derivation of Relation (3)

First of all, we will try to establish the relation (3). Taking into account that the series on the right-hand side of (3) is uniformly convergent, we can rewrite it as follows:

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n^{\lambda}}J_{k+\lambda}(n\alpha)\frac{\sin\alpha(m-n)}{m-n}=\frac{\alpha}{2\pi}\int_{-1}^{1}e^{i\alpha m\eta}\Phi_k(\eta,\alpha)d\eta \tag{4}$$

where

$$\Phi_k(\eta,\alpha)=\sum_{n=-\infty}^{\infty}\frac{1}{n^{\lambda}}J_{k+\lambda}(n\alpha)e^{-i\alpha n\eta} \tag{5}$$

According to Hönl et al. [7, p. 435], the following formula holds

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_\nu(\epsilon x)}{x^{\mu}}\frac{\sin[\epsilon(x-y)]}{x-y}dx=\frac{J_\nu(\epsilon y)}{y^{\mu}} \tag{6}$$

for $0\le\mu\le\nu$, $\nu=0,1,2\ \ldots$, with ϵ being a real parameter. We now assume that (6) is valid for any μ and ν such that $0\le\mu\le\nu$. Then, by setting

$$\nu=k+\lambda,\quad \mu=\lambda,\quad y=n \tag{7}$$

and substituting (6) in (5), we obtain

$$\Phi_k(\eta,\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_{k+\lambda}(\alpha x)}{x^{\lambda}}\sum_{n=-\infty}^{\infty}e^{-i\alpha n\eta}\frac{\sin\alpha(x-n)}{x-n}dx \tag{8}$$

It is easy to show that the series in (8) can be given as

$$\sum_{n=-\infty}^{\infty}e^{-i\alpha n\eta}\frac{\sin\alpha(x-n)}{x-n}=\pi e^{-i\alpha x\eta},\quad |\eta|<1 \tag{9}$$

which, in turn, gives

$$\Phi_k(\eta,\alpha)=\int_{-\infty}^{\infty}\frac{J_{k+\lambda}(\alpha x)}{x^{\lambda}}e^{-i\alpha x\eta}dx \tag{10}$$

Now let us substitute (10) into (4) and integrate the result with respect to η over $\eta\in(-1,1)$. This yields

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n^{\lambda}}J_{k+\lambda}(n\alpha)\frac{\sin\alpha(m-n)}{m-n}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_{k+\lambda}(x\alpha)}{x^{\lambda}}\frac{\sin\alpha(x-m)}{x-m}dx \tag{11}$$

Finally, we establish the desired result (3) by comparing (6) and (11). It should be noted that the relation (3) is exactly analogous to the integral representation (6). The equality (6) has been verified numerically (see Table 5).

2.2 An Alternative Derivation for Expression (1)

Returning to (1), we consider initially

$$\tilde{S}_k^\lambda = \sum_{n=-\infty}^{\infty} \frac{1}{n^\lambda} J_{k+\lambda}(n\alpha), \quad \lambda \geq 0 \tag{12}$$

Using the Gegenbauer integral representation for the Bessel function [8, p. 50]

$$\frac{J_{k+\lambda}(n\alpha)}{(n\alpha)^\lambda} = a_k^\lambda \int_{-1}^{1} e^{in\alpha\eta}(1-\eta^2)^{\lambda-\frac{1}{2}} C_k^\lambda(\eta) d\eta \tag{13}$$

where $\Re(\lambda) > -1/2$, and $C_k^\lambda(\eta)$ is the Gegenbaur Polinomials and

$$a_k^\lambda = \frac{(-i)^k}{2^\lambda} \frac{\Gamma(2\lambda)\Gamma(k+1)}{\Gamma(\lambda+\frac{1}{2})\Gamma(k+2\lambda)} \tag{14}$$

with $\Gamma(x)$ being the Gamma function, we can rewrite (12) as

$$\tilde{S}_k^\lambda = \alpha^\lambda a_k^\lambda \int_{-1}^{1} \sum_{n=-\infty}^{\infty} e^{in\alpha\eta}(1-\eta^2)^{\lambda-\frac{1}{2}} C_k^\lambda(\eta) d\eta \tag{15}$$

In writing (15), we have used the fact that the series in (12) is uniformly convergent. If we replace the sum in (15) with

$$\sum_{n=-\infty}^{\infty} e^{in\alpha\eta} = \frac{2\pi}{\alpha}\delta(\eta) \tag{16}$$

where $\delta(\eta)$ is the Dirack delta function, then the following formula holds:

$$\tilde{S}_k^\lambda = \alpha^\lambda a_k^\lambda \frac{2\pi}{\alpha} C_k^\lambda(0) \tag{17}$$

where the Gegenbauer Polynomial $C_k^\lambda(0)$, is given [9, p. 777] by

$$C_k^\lambda(0) = \begin{cases} 0, & k = 2m+1 \\ \dfrac{(-1)^{k/2}\Gamma\left(\frac{k}{2}+\lambda\right)}{\Gamma(\lambda)\Gamma\left(\frac{k}{2}+1\right)}, & k = 2m \end{cases} \tag{18}$$

We now substitute (18) into (17), and simplify the result. This yields

$$\tilde{S}_k^\lambda = \left(\frac{\alpha}{2}\right)^{\lambda-1} \frac{[1+(-1)^k]\Gamma\left(\frac{k+1}{2}\right)}{2\Gamma\left(\frac{k+1}{2}+\lambda\right)} \tag{19}$$

It's easy to show that

$$\tilde{S}_k^\lambda = \left(\frac{\alpha}{2}\right)^\lambda \frac{\delta_{k,0}}{\Gamma(\lambda+1)} + [1+(-1)^k] S_k^\lambda, \quad k \geq 0 \tag{20}$$

where $\delta_{k,0} = 1$ if $k = 0$ and 0 if $k \neq 0$. Using (19) and (20), we obtain

$$S_k^\lambda = \left(\frac{\alpha}{2}\right)^{\lambda-1} \frac{\Gamma\left(\frac{k+1}{2}\right)}{2\Gamma\left(\frac{k+1}{2}+\lambda\right)} - \left(\frac{\alpha}{2}\right)^{\lambda} \frac{\delta_{k,0}}{2\Gamma(\lambda+1)}, \quad \lambda \geq 0 \tag{21}$$

We have shown that the Schlömilch type series (1) can be given in the explicit form of (21). Again, in the reference book [6, p.678], we find only special cases of (21) when λ is an integer. It should be emphasized that the representation (21) is superior to (1) in terms of computational efficiency. This point is illustrated in Table 1 for different values of λ, *alpha*, and N where N indicates the number of terms used in calculating the sum given by (1). All of the numerical simulations in this paper are run on a 486 DX2 IBM compatible PC. The values in the last column of Table 1 are obtained using (21). Each cell of the *Computation Time* row shows the CPU time used in computing the values in the respective columns as a whole. We see from Table 1 that a great reduction in computation time is established by using (21).

	Eq. (1), λ= 0.6, α= 0.1			Eq. (21)
k	N			Exact
	100	200	500	
0	3.140474	2.954448	2.969827	2.994830
1	1.926517	1.790110	1.864076	1.854727
2	1.276188	1.433947	1.429102	1.403439
3	1.018130	1.232059	1.152026	1.159204
4	1.037893	0.997560	0.975689	1.002457
5	1.079373	0.814917	0.894016	0.891696
Computation Time (s)	1.20	3.57	15.71	0.000000

	Eq. (1), λ= 0.9, α= 0.2			Eq. (21)
k	N			Exact
	100	200	500	
0	1.180201	1.192931	1.191302	1.192006
1	0.645554	0.648681	0.655206	0.654486
2	0.459430	0.447669	0.449853	0.449091
3	0.355926	0.349517	0.343222	0.344467
4	0.274589	0.283094	0.279823	0.280682
5	0.222861	0.233091	0.238744	0.237563
Computation Time (s)	1.70	5.49	20.10	0.000000

Table 1. Comparison of Formulae (1) and (21) for different values of λ and α with computation time.

2.3 Some Special Cases for the Series of (1)

Below are some examples of special cases of the series (1).

Case # 1: $\lambda = 0$

$$S_k^0 = \sum_{n=1}^{\infty} J_k(n\alpha) = \frac{1}{\alpha} - \frac{\delta_{k,0}}{2} \tag{22}$$

Case # 2: $\lambda = 1$

$$S_k^1 = \sum_{n=1}^{\infty} \frac{1}{n} J_{k+1}(n\alpha) = \frac{1}{k+1} - \frac{\alpha}{4}\delta_{k,0} \tag{23}$$

Case # 3: $\lambda = 2$

$$S_k^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} J_{k+2}(n\alpha) = \frac{\alpha}{(k+3)(k+1)} - \frac{\alpha^2}{16}\delta_{k,0} \tag{24}$$

Case # 4: $k = 0$

$$S_0^\lambda = \sum_{n=1}^{\infty} \frac{1}{n^\lambda} J_\lambda(n\alpha) = \left(\frac{\alpha}{2}\right)^{\lambda-1} \frac{\sqrt{\pi}}{2\Gamma\left(\lambda + \frac{1}{2}\right)} - \left(\frac{\alpha}{2}\right)^{\lambda} \frac{1}{2\Gamma(\lambda+1)} \tag{25}$$

If we use the fact that ([9, p. 257])

$$\frac{\Gamma(x+a)}{\Gamma(x+b)}_{x\to\infty} \sim x^{a-b} \tag{26}$$

then the asymptotic behavior of S_k^λ as $k \to \infty$ can be given as

$$S_{k\,k\to\infty}^\lambda \sim k^{-\lambda} \tag{27}$$

An alternative derivation only for (22) is also given by Twersky [10].

2.4 A Closed Form Representation for (2)

In this subsection, we try to obtain a more useful form for (2), which was earlier given by

$$\widetilde{V}_{k+\lambda}^{p+\mu}(\nu) = \sum_{n=-\infty}^{\infty} \frac{1}{n^\nu} J_{k+\lambda}(n\alpha) J_{p+\mu}(n\alpha) \tag{28}$$

where we consider the case

$$\nu = \lambda + \mu \tag{29}$$

Using the integral representation for the products of Bessel functions [8, p. 150] we can write

$$J_{k+\lambda}(n\alpha) J_{p+\mu}(n\alpha) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} J_{k+p+\lambda+\mu}(2n\alpha\cos\theta)\cos(k-p+\lambda-\mu)\theta d\theta \tag{30}$$

where $\Re(\lambda+\mu+k+p)>-1$. Now, let us substitute (29) and (30) into (28), in which case we obtain

$$\begin{aligned}\tilde{V}_{k+\lambda}^{p+\mu}(\lambda+\mu)&=\frac{2}{\pi}\int\limits_0^{\frac{\pi}{2}}\sum_{n=-\infty}^{\infty}\frac{J_{k+p+\lambda+\mu}(2n\alpha\cos\theta)}{n^{\lambda+\mu}}\cos(k-p+\lambda-\mu)\theta d\theta\\&=\frac{2}{\pi}\int\limits_0^{\frac{\pi}{2}}\tilde{S}_{k+p}^{\lambda+\mu}\cos(k-p+\lambda-\mu)\theta d\theta\end{aligned}\tag{31}$$

where, as we have shown before (see (19)),

$$\begin{aligned}\tilde{S}_{k+p}^{\lambda+\mu}&=\sum_{n=-\infty}^{\infty}\frac{1}{n^{\lambda+\mu}}J_{k+p+\lambda+\mu}(2n\alpha\cos\theta)\\&=[1+(-1)^{k+p}](\alpha\cos\theta)^{\lambda+\mu-1}\frac{\Gamma\left(\frac{k+p+1}{2}\right)}{2\Gamma\left(\frac{k+p+1}{2}+\lambda+\mu\right)}\end{aligned}\tag{32}$$

Using (32) in (31) the following equation is obtained

$$\begin{aligned}\tilde{V}_{k+\lambda}^{p+\mu}(\lambda+\mu)=&\frac{\alpha^{\lambda+\mu-1}\Gamma\left(\frac{k+p+1}{2}\right)}{\pi\Gamma\left(\frac{k+p+1}{2}\right)+\lambda+\mu)}\\&\times[1+(-1)^{k+p}]\int_0^{\frac{\pi}{2}}\cos^{\lambda+\mu-1}\theta\cos(k-p+\lambda-\mu)\theta d\theta\end{aligned}\tag{33}$$

The integral on the right-hand side of (33) can be given ([11, p.372]) as

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\cos^{\lambda+\mu-1}\theta\cos(k-p+\lambda-\mu)\theta d\theta\\&\qquad=\frac{\pi\Gamma(\lambda+\mu)}{2^{\lambda+\mu}\Gamma\left(\frac{k-p+1}{2}+\lambda\right)\Gamma\left(\frac{p-k+1}{2}+\mu\right)},\quad\lambda+\mu>0\end{aligned}\tag{34}$$

Now, by substituting (34) into (33), we have

$$\begin{aligned}\tilde{V}_{k+\lambda}^{p+\mu}(\lambda+\mu)=&\left(\frac{\alpha}{2}\right)^{\lambda+\mu-1}\\&\times\frac{[1+(-1)^{k+p}]\Gamma(\lambda+\mu)\Gamma\left(\frac{k+p+1}{2}\right)}{2\Gamma\left(\frac{k+p+1}{2}+\lambda+\mu\right)\Gamma\left(\frac{k-p+1}{2}+\lambda\right)\Gamma\left(\frac{p-k+1}{2}+\mu\right)},\quad\lambda+\mu>0\end{aligned}\tag{35}$$

It's easy to show that

$$\tilde{V}_{k+\lambda}^{p+\mu}(\lambda+\mu)=\left(\frac{\alpha}{2}\right)^{\lambda+\mu}\frac{\delta_{k,0}\delta_{p,0}}{\Gamma(\lambda+1)\Gamma(\mu+1)}+[1+(-1)^{k+p}]\;V_{k+\lambda}^{p+\mu}(\lambda+\mu)\tag{36}$$

Thus, we have

$$V_{k+\lambda}^{p+\mu}(\lambda+\mu) = \left(\frac{\alpha}{2}\right)^{\lambda+\mu-1} \times \frac{\Gamma(\lambda+\mu)\Gamma\left(\frac{k+p+1}{2}\right)}{2\Gamma\left(\frac{k+p+1}{2}+\lambda+\mu\right)\Gamma\left(\frac{k-p+1}{2}+\lambda\right)\Gamma\left(\frac{p-k+1}{2}+\mu\right)} - \left(\frac{\alpha}{2}\right)^{\lambda+\mu}\frac{\delta_{k0}\delta_{p0}}{2\Gamma(\lambda+1)\Gamma(\mu+1)}, \quad \Re(\lambda+\mu)>0 \tag{37}$$

Only special cases of our representation (37) when $k=p=0$, are found in the reference book [6, p.683]. It should be noted that our results (21) and (37) for the series (1) and (2) are analogous to the well known, so called discontinuous Weber-Schafheitlin's integral representation for the Bessel functions [8, p. 403; 6, p. 211]. Indeed, these have the form [6, p.174, 211]

$$\int_0^\infty \frac{J_{k+\lambda}(\alpha x)}{x^\lambda}dx = \left(\frac{\alpha}{2}\right)^{\lambda-1}\frac{\Gamma\left(\frac{k+1}{2}\right)}{2\Gamma\left(\frac{k+1}{2}+\lambda\right)}, \quad \Re(\lambda)\geq 0 \tag{38}$$

and

$$\int_0^\infty \frac{J_{k+\lambda}(\alpha x)J_{p+\mu}(\alpha x)}{x^{\lambda+\mu}}dx = \left(\frac{\alpha}{2}\right)^{\lambda+\mu-1}\frac{\Gamma\left(\frac{k+p+1}{2}\right)\Gamma(\lambda+\mu)}{2\Gamma\left(\frac{k+p+1}{2}+\lambda+\mu\right)\Gamma\left(\frac{k-p+1}{2}+\lambda\right)\Gamma\left(\frac{p-k+1}{2}+\mu\right)} \tag{39}$$

where $\Re(\lambda+\mu)>0$. A quick comparison of the representations (21) and (37) respectively with (38) and (39) shows that we have established representations for the series (1) and (2) which are analogous to the integral representations (38) and (39). Using (27) we obtain the asymptotic limits for $V_{k+\lambda}^{p+\mu}(\lambda+\mu)$ as follows:

$$V_{k+\lambda}^{k+\mu}(\lambda+\mu)_{k\to\infty} \sim k^{-(\lambda+\mu)};$$

$$V_{k+\lambda}^{p+\mu}(\lambda+\mu)_{k\to\infty} \sim k^{-2(\lambda+\mu)}; \tag{40}$$

$$V_{k+\lambda}^{p+\mu}(\lambda+\mu)_{p\to\infty} \sim p^{-2(\lambda+\mu)}$$

Numerous special cases of interest are obtained by giving special values to the constants λ,μ in (37). The following are some of the important special cases.

Case # 1: $\lambda=0$, $\mu=1$

$$V_k^{p+1}(1) = \frac{2\cos\pi\left(\frac{k-p}{2}\right)}{\pi(k+p+1)(p-k+1)} - \frac{\alpha}{4}\delta_{k0}\delta_{p0} \tag{41}$$

Case # 2: $\lambda = 1\,,\ \mu = 0$

$$V_{k+1}^{p}(1) = \frac{2\cos\pi\left(\frac{k-p}{2}\right)}{\pi(k+p+1)(k-p+1)} - \frac{\alpha}{4}\delta_{k0}\delta_{p0} \tag{42}$$

Case # 3: $\lambda = \mu = 1$

$$V_{k+1}^{p+1}(2) = \frac{4\alpha\cos\pi\left(\frac{k-p}{2}\right)}{\pi(k+p+3)(k+p+1)[1-(k-p)^2]} - \frac{\alpha^2}{8}\delta_{k0}\delta_{p0} \tag{43}$$

In obtaining the special cases above, we made use of the fact that ([9, p. 256])

$$\Gamma\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}+x\right) = \frac{\pi}{\cos\pi x} \tag{44}$$

A comparison of (2) and (37) in terms of computational efficiency is given in Tables 2 and 3 for different values of α, λ, and μ. The computation times are given for the whole $k \times p$ matrices calculated for different N. Again, it is clear from Tables 2 and 3 that representation (37) greatly reduces the computation time. Hence, we can say that representation (37) is far more efficient than (2) from a numerical point of view.

		Eq. (2) α= 0.1, μ= 0.6, λ= 0.7							Computation time (s)
			p						
		k	0	1	2	3	4	5	
N	100	0	0.379159	0.133901	0.026231	-0.008133	-0.008931	-0.001942	14.34
		1	0.116451	0.104794	0.056893	0.015904	-0.002970	-0.005622	
		2	0.015679	0.050277	0.052687	0.033597	0.012731	0.000286	
		3	-0.010599	0.010575	0.030207	0.033160	0.022858	0.009900	
		4	-0.007793	-0.004936	0.009551	0.020941	0.021904	0.015553	
		5	-0.000576	-0.005413	-0.001274	0.007906	0.014430	0.015198	
	200	0	0.382911	0.134441	0.022590	-0.010143	-0.006390	0.001885	42.85
		1	0.115922	0.108161	0.058211	0.013087	-0.005731	-0.004750	
		2	0.011772	0.050576	0.056688	0.034926	0.009471	-0.003386	
		3	-0.011711	0.007365	0.030579	0.036533	0.024243	0.007447	
		4	-0.004522	-0.006944	0.005781	0.021419	0.025903	0.017927	
		5	0.002923	-0.003667	-0.004412	0.004833	0.015959	0.019512	
	500	0	0.384707	0.134810	0.020837	-0.010841	-0.004825	0.003096	186.97
		1	0.115764	0.109794	0.058553	0.011516	-0.006467	-0.003423	
		2	0.009958	0.050401	0.058481	0.035437	0.007816	-0.004442	
		3	-0.011906	0.005697	0.030587	0.038207	0.024659	0.005904	
		4	-0.002770	-0.007202	0.003997	0.021344	0.027665	0.018582	
		5	0.003709	-0.002091	-0.005022	0.003130	0.016135	0.021289	
Eq. (37) α= 0.1, μ= 0.6, λ= 0.7									
		0	0.385447	0.134996	0.020100	-0.011005	-0.004102	0.003335	0.055
		1	0.115711	0.110551	0.058694	0.010768	-0.006670	-0.002699	
		2	0.009213	0.050309	0.059224	0.035572	0.007084	-0.004653	
		3	-0.011837	0.004935	0.030490	0.038963	0.024818	0.005166	
		4	-0.002019	-0.007174	0.003247	0.021272	0.028410	0.018730	
		5	0.003720	-0.001328	-0.005005	0.002368	0.016055	0.022043	

Table 2. Comparison of Formulae (2) and (37) for $\alpha = 0.1$, $\mu = 0.6$, and $\lambda = 0.7$ with computation time.

		Eq. (2) α= 0.2, μ= 0.7, λ= 0.5							Computation time (s)
			p						
		k	0	1	2	3	4	5	
N	100	0	0.571088	0.164204	0.005243	-0.022386	-0.005287	0.006767	21.53
		1	0.226266	0.176111	0.075591	0.005092	-0.014017	-0.005125	
		2	0.041172	0.102139	0.095424	0.047349	0.005277	-0.009345	
		3	-0.018259	0.024767	0.062824	0.062951	0.034231	0.005264	
		4	-0.012483	-0.010390	0.018282	0.044460	0.045520	0.026094	
		5	0.003180	-0.009417	-0.006039	0.014648	0.033321	0.034744	
	200	0	0.574070	0.163516	0.002177	-0.022326	-0.002222	0.007797	67.30
		1	0.227356	0.178782	0.074837	0.002263	-0.014046	-0.002305	
		2	0.038316	0.103126	0.098403	0.046972	0.002209	-0.010058	
		3	-0.019892	0.022277	0.064142	0.065719	0.033680	0.002291	
		4	-0.010072	-0.012024	0.015664	0.045563	0.048436	0.026021	
		5	0.005627	-0.007475	-0.008243	0.012241	0.034857	0.037717	
	500	0	0.575494	0.163087	0.000727	-0.022035	-0.000736	0.007750	234.34
		1	0.227861	0.180267	0.074424	0.000739	-0.013834	-0.000748	
		2	0.036919	0.103634	0.099831	0.046599	0.000733	-0.009928	
		3	-0.020519	0.020836	0.064681	0.067211	0.033339	0.000744	
		4	-0.008754	-0.012712	0.014303	0.046123	0.049871	0.025697	
		5	0.006460	-0.006147	-0.008995	0.010840	0.035424	0.039213	
Eq. (37) α= 0.2, μ= 0.7, λ= 0.5									
		0	0.576217	0.163238	0.000000	-0.021823	0.000000	0.007578	0.055
		1	0.228534	0.180998	0.074199	0.000000	-0.013640	0.000000	
		2	0.036200	0.103879	0.100555	0.046375	0.000000	-0.009743	
		3	-0.020776	0.020111	0.064924	0.067942	0.033125	0.000000	
		4	-0.008044	-0.012985	0.013588	0.046375	0.050595	0.025481	
		5	0.006752	-0.005435	-0.009275	0.010119	0.035673	0.039944	

Table 3. Comparison of Formulae (2) and (37) for $\alpha = 0.2$, $\mu = 0.7$, and $\lambda = 0.5$ with computation time.

III. ELEMENTARY FUNCTION REPRESENTATON

Finally, we will try to find an alternative representation for the series (2) in terms of more of an elementary function, when

$$\nu = \lambda + \mu + 1 \tag{45}$$

We will further assume that, in (2)

$$k + p = \text{even} > 0 \tag{46}$$

The conditions (45) and (46) are just what arise in a variety of diffraction problems [1–5]. Alternative derivations of (2) for the case of $\mu = \lambda$ and $\nu = 2\lambda + 1$ are given in [4]. The present derivation is more general. We will again make use of the integral representation (30) for the Bessel function, and the Sonine's first finite integral representation [8, p. 373], i.e.,

$$\begin{aligned} J_{k+p+\mu+\lambda}(2n\alpha\cos\theta) = \frac{(2n\alpha\cos\theta)^{\lambda+\mu}}{2^{\lambda+\mu-1}\Gamma(\lambda+\mu)} \int_0^{\frac{\pi}{2}} J_{k+p}(2n\alpha\cos\theta\sin\phi)\times \\ \times \sin^{k+p+1}\phi\cos^{2\lambda+2\mu-1}\phi d\phi, \quad \Re(\lambda+\mu) > 0 \end{aligned} \tag{47}$$

At this point, let us use $k + p =$ even, and the following integral representation for $J_{k+p}(x)$ [8, p. 21]

$$J_{k+p}(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)\cos(k+p)\phi d\phi \tag{48}$$

Substituting (48) into (47), and later (47) into (30), we obtain

$$J_{k+\lambda}(n\alpha)J_{p+\mu}(n\alpha) = \frac{4}{\pi^2}\frac{(n\alpha)^{\lambda+\mu}}{\Gamma(\lambda+\mu)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\cos(k-p+\lambda-\mu)\theta\cos^{\lambda+\mu}\theta$$
$$\times\{\sin^{k+p+1}\phi\cos^{2\lambda+2\mu-1}\phi[\cos(2n\alpha\cos\theta\sin\phi\sin\psi)\cos(k+p)\psi]\}d\theta d\phi d\psi \tag{49}$$

Substituting the integral representation (49) into (2) with $\nu = \lambda+\mu+1$, we obtain

$$V_{k+\lambda}^{p+\mu}(\lambda+\mu+1) = -\frac{4}{\pi^2}\frac{\alpha^{\lambda+\mu}}{\Gamma(\lambda+\mu)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\cos(k+p+\lambda-\mu)\theta\cos^{\lambda+\mu}\theta$$
$$\times\{\sin^{k+p+1}\phi\cos^{2\lambda+2\mu-1}\phi\cos(k+p)\psi\ln 2\sin(\alpha\cos\theta\sin\phi\sin\psi)\}d\theta d\phi d\psi \tag{50}$$

In deriving (50), we made use of the fact that ([11, p. 38])

$$\sum_{n=1}^{\infty}\frac{1}{n}\cos(2n\alpha\cos\theta\sin\phi\sin\Psi) = -\ln 2\sin(\alpha\cos\theta\sin\phi\sin\Psi) \tag{51}$$

Using the series representation for the function $\ln(2\sin x)$ as ([11, p. 46])

$$\ln(2\sin x) = \ln 2x - \sum_{s=1}^{\infty}\frac{\zeta(2s)}{s}\left(\frac{x}{\pi}\right)^{2s} \tag{52}$$

where $\zeta(2s)$ is the Riemann Zeta function [9, p. 807], we can represent the $V_{k+\lambda}^{p+\mu}(\lambda+\mu+1)$ as follows:

$$V_{k+\lambda}^{p+\mu}(\lambda+\mu+1) = Q_{kp}^{(1)} + Q_{kp}^{(2)} \tag{53}$$

where

$$Q_{kp}^{(1)} = -\frac{4}{\pi^2}\frac{\alpha^{\mu+\lambda}}{\Gamma(\lambda+\mu)}[A_0^+B_0^+C_0^+\ln(2\alpha) + C_0^+B_0^+A_0^- + C_0^+A_0^+B_0^- + A_0^+B_0^+C_0^-] \tag{54}$$

and

$$Q_{kp}^{(2)} = \frac{4}{\pi^2}\frac{\alpha^{\lambda+\mu}}{\Gamma(\lambda+\mu)}\sum_{s=1}^{\infty}\frac{\zeta(2s)}{s}\left(\frac{\alpha}{\pi}\right)^{2s}A_{2s}^+B_{2s}^+C_{2s}^+ \tag{55}$$

with

$$A_s^{\pm} = \int_0^{\frac{\pi}{2}}\left\{\begin{matrix}1\\ \ln\cos\theta\end{matrix}\right\}\cos(k-p+\lambda-\mu)\theta\cos^{\lambda+\mu+s}\theta d\theta \tag{56}$$

$$B_s^{\pm} = \int_0^{\frac{\pi}{2}}\left\{\begin{matrix}1\\ \ln\sin\varphi\end{matrix}\right\}\sin^{(s+k+p+1)}\varphi\cos^{2\lambda+2\mu-1}\varphi d\varphi \tag{57}$$

$$C_s^{\pm} = \int_0^{\pi} \left\{ \begin{matrix} 1 \\ \ln \sin \psi \end{matrix} \right\} \cos(k+p)\psi \sin^s \psi d\psi \tag{58}$$

The integrals in (56)–(58) are the table integrals [11], and can be calculated in analytical form (see the Appendix). As a result, we obtain

$$Q_{kp}^{(1)} = \frac{Q_{kp}^{\lambda\mu}(0)}{k+p}$$

$$= \left(\frac{\alpha}{2}\right)^{\lambda+\mu} \frac{\Gamma(\lambda+\mu+1)\Gamma\left(\frac{k+p}{2}\right)}{2\Gamma\left(\frac{k-p}{2}+\lambda+1\right)\Gamma\left(\frac{p-k}{2}+\mu+1\right)\Gamma\left(\frac{k+p}{2}+\lambda+\mu+1\right)}, \quad k+p>0 \tag{59}$$

$$Q_{kp}^{(2)} = \frac{1}{\pi}\sum_{s=1}^{\infty}\left(\frac{\alpha}{2\pi}\right)^{2s}\frac{\zeta(2s)}{s}Q_{kp}^{\mu\lambda}(s)C_{2s}^{+} \tag{60}$$

where

$$Q_{kp}^{\lambda\mu}(s)$$

$$= \left(\frac{\alpha}{2}\right)^{\lambda+\mu} \frac{\Gamma(2s+\lambda+\mu+1)\Gamma\left(s+1+\frac{k+p}{2}\right)}{\Gamma\left(\frac{k-p}{2}+\lambda+s+1\right)\Gamma\left(\frac{p-k}{2}+\mu+s+1\right)\Gamma\left(\frac{k+p}{2}+\lambda+\mu+s+1\right)} \tag{61}$$

Eq. (6), λ=μ= 1/2 α=π/2						
	p					
k	0	1	2	3	4	5
0	$1.75\ 10^{-3}$	0	$2.14\ 10^{-4}$	0	$-3.97\ 10^{-6}$	0
1	0	$3.13\ 10^{-4}$	0	$-9.04\ 10^{-6}$	0	$2.4\ 10^{-7}$
2	2.14 10-4	0	1.17 10-5	0	4.47 10-7	0
3	0	$-9.04\ 10^{-6}$	0	$5.47\ 10^{-7}$	0	$-2.38\ 10^{-8}$
4	$3.97\ 10^{-6}$	0	$4.47\ 10^{-7}$	0	$2.81\ 10^{-8}$	0
5	0	$2.4\ 10^{-7}$	0	$2.38\ 10^{-8}$	0	$1.53\ 10^{-9}$
Eq. (60), λ=μ=1/2 α=π/10						
0	$-6.5\ 10^{-4}$	0	$2.7\ 10^{-7}$	0	$-1.6\ 10^{-8}$	0
1	0	$4.0\ 10^{-7}$	0	$4.0\ 10^{-10}$	0	$3.4\ 10^{-13}$
2	$2.7\ 10^{-7}$	0	$5.3\ 10^{-10}$	0	$6.9\ 10^{-13}$	0
3	0	$4.0\ 10^{-10}$	0	$8.6\ 10^{-13}$	0	$1.3\ 10^{-15}$
4	$-1.6\ 10^{-8}$	0	$6.9\ 10^{-13}$	0	$-1.5\ 10^{-15}$	0
5	0	$3.4\ 10^{-13}$	0	$1.3\ 10^{-15}$	0	$3.0\ 10^{-18}$

Table 4. Values of Equation (60) for different values of α.

The expression for C_{2s}^{+} presented in the Appendix (A. 3) Using (26) and (44) we may estimate the following asymptotic limits for $V_{k+\lambda}^{\mu+1}(\lambda+\mu+1)$:

$$\begin{aligned} V_{k+\lambda}^{k+\mu}(\lambda+\mu+1)_{k\to\infty} &\sim Q_{kk\,k\to\infty}^{(1)} \sim k^{-\lambda-\mu-1} \\ V_{k+\lambda}^{p+\mu}(\lambda+\mu+1)_{k\to\infty} &\sim Q_{kp}^{(1)} \sim k^{-2(\lambda+\mu)-2} \\ V_{k+\lambda}^{p+\mu}(\lambda+\mu+1)_{p\to\infty} &\sim Q_{kp}^{(1)} \sim p^{-2(\lambda+\mu)-2} \end{aligned} \tag{62}$$

In the same fashion , we can find the representation for $V_{k+\lambda}^{p+\mu}(\lambda+\mu+1)$ when $k=p=0, \lambda=\mu$ as follows:

$$V_{\lambda}^{\mu}\,|_{\lambda=\mu} = V_{\lambda}^{\lambda}\,(2\lambda+1) = Q_{00}^{(1)} + Q_{00}^{(2)} \tag{63}$$

where

$$Q_{00}^{(1)} = -\left(\frac{\alpha}{2}\right)^{2\lambda}\frac{1}{2\Gamma^2\,(\lambda+1)}\{2\ln\alpha+\psi\,(\lambda+1/2)-\psi\,(\lambda+1)+\psi\,(1)-\psi\,(2\lambda+1)\} \tag{64}$$

$$Q_{00}^{(2)} = \frac{1}{\pi}\sum_{s=1}^{\infty}\left(\frac{\alpha}{2\pi}\right)^{2s}\frac{\varsigma\,(2s)}{s}Q_{00}^{\lambda\lambda}\,(s)\,C_{2s}^{+}\,|_{k=p=0} \tag{65}$$

with $\psi\,(x)$ is the *psi* function. It is very important to note that, as our numerical experiments have shown, when $\alpha \leq \frac{\pi}{2}$ the leading terms of $V_{k+\lambda}^{p+\mu}(\lambda+\mu+1)$ are given by

$$V_{k+\lambda}^{p+\mu}(\lambda+\mu+1) = Q_{kp}^{(1)} + O(10^{-4}) \tag{66}$$

which means that $Q_{kp}^{(2)} = O(10^{-4})$. Thus, in this case we can say that we have the analytical representation for $V_{k+\lambda}^{p+\mu}(\lambda+\mu+1)$. Our numerical experiments also show that the series in (60) has a very fast convergence rate for $\alpha \leq \pi$. The following are some of the important special cases in the application of our representation to the solution of various diffraction problems:

Case #1: $\lambda=\mu=0$

$$\begin{aligned} Q_{kp}^{(1)} &= \frac{2\sin\pi\frac{k-p}{2}}{\pi\;k^2-p^2}\frac{1}{\left(\frac{p-k}{2}+1\right)\left(\frac{p+k}{2}+1\right)} \\ &= \begin{cases} \dfrac{1}{2k} & k=p \\ 0 & k\neq p,\ k+p=\text{even} \end{cases} \end{aligned} \tag{67}$$

$$Q_{00}^{(1)} = -\ln\left(\frac{\alpha}{2}\right) \tag{68}$$

Case #2: $\lambda = 0\,,\ \mu = 1$

$$Q_{kp}^{(1)} = \alpha \frac{\sin \pi \frac{k-p}{2}}{k^2 - p^2} \frac{1}{\left(\frac{p-k}{2}+1\right)\left(\frac{p+k}{2}+1\right)}$$

$$= \begin{cases} \dfrac{\alpha}{4k(k+1)} & k = p \\ 0 & k \neq p,\ k+p = \text{even} \end{cases} \tag{69}$$

		$\frac{1}{\pi}\sum_{n=-NN}^{NN} \frac{1}{n^{\lambda}} J_{k+\lambda}(n\alpha) \frac{\sin \alpha(m-n)}{m-n}$				
$\alpha=\pi/2=$ 1.570796	$\frac{J_{k+\lambda}(m\alpha)}{m^{\lambda}}$	NN=5	NN=10	NN=50	NN=100	NN=500
k=1, λ=1, m=1	2.497016e-01	2.400811e-01	2.469212e-01	2.494357e-01	2.496069e-01	2.496931e-01
k=3, λ=1, m=1	1.399604e-02	2.573342e-02	1.721381e-02	1.427329e-02	1.409291e-02	1.400461e-02
k=5, λ=1, m=1	2.983476e-04	-8.549538e-03	-3.266610e-03	3.746473e-06	1.981240e-04	2.897138e-04
k=1, λ=1, m=5	-3.009402e-02	-5.012642e-02	-3.314727e-02	-3.036101e-02	-3.018889e-02	-3.010255e-02
k=3, λ=1, m=5	-1.389344e-02	1.079264e-02	-1.034994e-02	-1.361507e-02	-1.379647e-02	-1.388487e-02
k=5, λ=1, m=5	6.918731e-02	5.379513e-02	6.526445e-02	6.889150e-02	6.908698e-02	6.917867e-02

Table 5. Values of the truncated series (3) with various values of the truncation parameter.

IV. CONCLUSION

We have shown that the Schlömilch type series with Bessel functions as in (1) and (2) can be represented in terms of more of elementary functions or an analytical form. Various special cases of (1) and (2) arise in different diffraction problems. Our results are also interesting from a mathematical point of view. They can be considered as the Weber-Schafheitlin representation for the Schlomilch type series (1) and (2). It should be noted that only special cases of our representation are given in the mathematical reference books.

APPENDIX

Evaluation of $A_s^{\pm}$, $B_s^{\pm}$, $C_s^{\pm}$

1. The expression for $A_s^{\pm}$ has the form [11, p. 372, 587]

$$A_s^{\pm} = \begin{cases} \dfrac{\pi\Gamma(s+\lambda+\mu+1)}{2^{s+\lambda+\mu+1}\Gamma\left(\frac{k-p}{2}+\lambda+s+1\right)\Gamma\left(\frac{p-k}{2}+\mu+s+1\right)}, & k+p \geq 0 \\ \frac{\sqrt{\pi}}{4}\frac{\Gamma(s+\lambda+\frac{1}{2})}{\Gamma(s+\lambda+1)}\left[\psi(\lambda+s+\frac{1}{2}) - \psi(s+\lambda+1)\right], & \lambda=\mu,\ k=p=0 \end{cases} \tag{A.1}$$

where $\psi(x)$ is the *psi* function. It should be noted that for any λ, μ and k, p the A_{2s}^{-} must be calculated only numerically

2. For $B_s^{\pm}$ we have [11, p. 369, 587]

$$B_s^{\pm} = \begin{cases} \dfrac{\Gamma(\lambda+\mu)\Gamma\left(\frac{k+p}{2}+s+1\right)}{2\Gamma\left(\frac{k+p}{2}+s+\lambda+\mu+1\right)}, & \lambda+\mu>0,\ k+p>0 \\ \frac{1}{4}\frac{\Gamma(s+1)\Gamma(\lambda+\mu)}{\Gamma(s+\lambda+\mu+1)}\left[\psi(s+1) - \psi(s+\lambda+\mu+1)\right], & \lambda+\mu>0,\ k=p=0 \end{cases} \tag{A.2}$$

3. For $C_s^{\pm}$ we have [11, p. 373, 584, 587]

$$C_{2s}^{+} = \begin{cases} \dfrac{\pi(-1)^{\frac{k+p}{2}}\Gamma(2s+1)}{2^{2s}\Gamma\left(s+1+\frac{k+p}{2}\right)\Gamma\left(s+1-\frac{k+p}{2}\right)}, & s \geq \frac{k+p}{2} \\ 0, & s < \frac{k+p}{2} \end{cases} \tag{A.3}$$

$$C_0^{+} = \begin{cases} \pi & k=p=0 \\ 0 & k+p>0 \end{cases} \tag{A.4}$$

$$C_0^{-} = \begin{cases} -\dfrac{\pi}{k+p} & k+p=\text{even}>0 \\ -\pi\ln 2 & k=p=0 \end{cases} \tag{A.5}$$

$$C_{2s}^{-} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s+1)}\left[\psi\left(s+\frac{1}{2}\right) - \psi(s+1)\right] \qquad k=p=0 \tag{A.6}$$

ACKNOWLEDGMENT

The first author of this paper is very grateful to the Osaka-Electrocommunication University and especially to M. Hashimoto for his kind hospitality.

REFERENCES

1. Veliev, E. I., and V. P. Shestopalov, "Wave diffraction by intersecting circular cylinder bodies," *Sov. Phys. Dokl.*, Vol. 30, 461–464, 1985.
2. Vavilov, V. N., E. I. Veliev, V. V. Veremey, and V. P. Shestopalov, "Effective method of solution of diffraction problems for waves incident on a cylindrical screen," *Sov. Phys. Dokl.*, Vol. 35, 640–644, 1990.
3. Vavilov, V. N., and E. I. Veliev, "Electromagnetic wave diffraction by cylindrical bodies with edges," *Electromagnetics*, Vol. 13, 339–357, 1993.
4. Gal'chenko, N. A., V. S. Mihalevskii, and G. P. Sinyavskii, "Waveguides of complex crossections," *Rostov-on-Don*, 1978 (in Russian).
5. Hongo, K., "Diffraction by a flanged parallel plate wave guide," *Radio Science*, Vol 7, 955–963, 1972.
6. Prudnikov, H. P., Y. H. Brychkov, and O. I. Marichev, *Integrals and Series, Vol. 2: Special Functions*, Gordon and Breach Science Publishers, 1986.
7. Hönl, H., A. W. Maue, and K. Westpfahl, *Theorie der Beugung*, Springer–Verlag, Berlin, 1961.
8. Watson, G. N., *A Treatiese on the Theory of Bessel Fuctions*, Cambridge University Press, 1966.
9. Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions*, New York, 1972.
10. Twersky, V., "Elementry Function Representations of Schlömilch Series," *Arch. Rational Mach. Anal.*, Vol. 8, 323–333, 1961.
11. Gradshteyn, I. S., and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 1980.

Eldar I. Veliev was born in Baku, Azerbaijan in 1951. He graduated from Azerbaijan State University in 1973. He received the Ph.D. and the Doctor of Science (Physics and Mathematics) degrees from Kharkov State University,, Kharkov, Ukraine in 1977 and 1988, respectively. He is a Leading Research Scientist at the Institute of Radiophysics and Electronics, Ukraninan Academy of Sciences, Kharkov, Ukraine. He is currently a Visiting Professor at the Department of Electrical and Electronics Engineering, Çukurova University, Adana, Turkey, His main research interests are diffraction theory, numerical-analytical methods in electromagnetic field theory, and theory of antennas.

Ali Okşaşoğlu was born in Turkey in 1960. He received the B.S. degree in Electrical Engineering from the Technical University of Istanbul, Turkey in 1983, and the M.S. and Ph.D. degrees in Electrical Engineering from the University of Arizona, Tucson, AZ, in 1987 and 1990, respectively. He is currently an Assistant Professor in the Department of Electrical and Electronics Engineering, Çukurova University, Adana, Turkey. His major research interests are linear and nonlinear circuit theory, analog filters, numerical analysis and chaos.