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MATHEMATICAL METHODS IN ELECTROMAGNETIC THEORY

APPLICATION OF FRACTIONAL OPERATORS TO DESCRIBING BOUNDARIES IN THE SCATTERING PROBLEMS *

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The possibility is analyzed for applying fractional operators in the problems of electromagnetic wave reflection from plane boundaries. The fractional derivative and fractional curl operator are considered, which are obtained as a result of fractionalization of the ordinary derivation and curl operators. The fractional curl operator can be used for describing the polarization reversal effect for the wave reflected from a biisotropic layer or boundary characterized by anisotropic impedance boundary conditions. The order of the fractional curl operator is determined through the constitutive parameters of the problem under consideration. Boundary conditions with a fractional derivative generalize the condition for perfectly electric and perfectly magnetic conducting boundaries. Application of the fractional boundary conditions (FBC) to modeling wave reflection from plane boundaries is analyzed. The scattering properties of a strip with FBC and of an impedance strip are compared by the example of the problem of diffraction at a strip of a finite width. Expressions have been derived which relate the fractional order with the impedance. It is shown that FBC can be used over a wide range of parameter variation for the wave reflection simulation from impedance boundaries, as well as from a dielectric layer. The FBC correspond to impedance boundaries with a pure imaginary value of the impedance. Also the specific features shown by the scattering characteristics of a strip with FBC, which are associated with its "superwave" properties, are analyzed.

KEY WORDS: fractional derivative, fractional curl operator, boundary condition, diffraction

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1. INTRODUCTION

Recently the fractional operators have found a wide range of application in various problems of electrodynamics. The fractional operators are defined as fractionalized form of the conventional ones. For example, fractional derivatives and integrals represent generalizations of ordinary derivatives and integrals. The fractional curl operator which is defined as a fractionalized form of the ordinary curl operator [1], is widely used as well.

The curl operator, curl^α, plays an important role in equations of electrodynamics, which allows constructing new solutions. Application of curl^α, with α being a fractional order, to a solution of the Maxwell equations $\left(\vec{E}^0, \vec{H}^0\right)$, where $\vec{E}^0$ and $\vec{H}^0$ are the electric and magnetic field strengths, respectively, in the form

$$\left(\vec{E}^\alpha, \vec{H}^\alpha\right) \equiv (ik_0)^{-\alpha} \text{curl}^\alpha \left(\vec{E}^0, \vec{H}^0\right) \tag{1}$$

results in that the fractional field $\left(\vec{E}^\alpha, \vec{H}^\alpha\right)$ is again a solution of the Maxwell's equations for the same medium as for the primary field $\left(\vec{E}^0, \vec{H}^0\right)$ [1]. Application of curl^α allows describing the polarization reversal effect for the field [2,3]. The fractional field $\left(\vec{E}^\alpha, \vec{H}^\alpha\right)$ serves as an intermediate solution between the primary $\left(\vec{E}^0, \vec{H}^0\right)$ and dual $\left(\vec{E}^1, \vec{H}^1\right) = \left(\eta_0 \vec{H}^0, -1/\eta_0 \vec{E}^0\right)$ solutions [4], which correspond to the fractional order values $\alpha = 0$ and $\alpha = 1$, respectively.

Making use of the operator curl^α for an example of the reflection problem for a plane wave normally incident upon an interface, Engheta [1] obtained an impedance boundary with the impedance equal to $\eta_\alpha = i\tan(\pi\alpha/2)$. This case corresponds to an intermediate situation between the perfectly electric and perfectly magnetic conducting boundaries under change α from 0 to 1.

In the present paper, 2D problems of electromagnetic wave reflection from an interface are treated. The solution is represented as a result of applying the fractional curl operator to the known solution of the impedance boundary problem. The new fractional solution Eq. (1) corresponds to a new boundary whose parameters are dependent on the primary boundary characteristics and the order of the fractional curl operator. By analyzing the inverse problem it is possible to determine the boundary characteristics from the known fractional field. To do so it is necessary to derive an equation relating α and constitutive parameters. This approach allows obtaining the solution to the reflection problem without actually solving the problem of wave reflection from a surface of a complex geometry. Instead, the curl^α operator with a specified α is applied to the familiar solution of the reflection problem for a simple boundary, normally either perfectly electric conducting or impedance one.

Construction of the simple boundary conditions to analyze the reflecting properties of composite materials represents an important problem. Two aspects are known in the consideration of the boundary conditions. The first one is associated with constructing boundary conditions to describe the scattering properties of a given physical structure. The second aspect relates to the determination of a physical structure with the specified boundary conditions. The latter problem is of great importance for the antenna theory, in particular, for designing antennas with the prescribed radiation characteristics.

On the other hand application of the new boundary conditions allows constructing model boundaries with new properties. Implementation of boundaries with arbitrary magnitudes of the parameters, which determine the boundary conditions, represents not a simple problem.

Boundary conditions should allow constructing an efficient numerical algorithm for obtaining the exact solution to within a specified accuracy. Construction of simple and adequate mathematical models for describing the scattering properties of surfaces is among general problems in the theory of diffraction.

A rather well studied boundary which occupies an intermediate position between the perfectly electric and perfectly magnetic conducting boundaries is represented by the impedance boundary [5,6],

$$\vec{n}\times\vec{E}(\vec{r})=\eta\vec{n}\times\left(\vec{n}\times\vec{H}(\vec{r})\right),\ \text{with}\ \vec{r}\in S, \tag{2}$$

where $\vec{n}$ is a normal to the surface S. The impedance assumes values from 0 for the perfectly electric conducting boundary to $i\infty$ for the perfectly magnetic conducting one.

The problem of diffraction by impedance boundaries was treated in a number of papers. The impedance boundary conditions were used successfully for modeling the reflecting properties of well-conducting materials, as well as gratings etc. In each case formulas exist for expressing the impedance as a function of the metal conductivity, grating parameters etc. The impedance boundary conditions are a kind of approximate boundary conditions, which have a limited range of applicability and cannot be used for describing the reflecting properties of the whole variety of surfaces.

The impedance boundary conditions can be further more accurately specified using derivatives of higher (integer) orders or generalized boundary conditions [5,7,8]. The general methodology of constructing exact impedance boundary conditions of a high order was presented by Hoppe and Rahmat-Samii [7]. The writers considered planar coverings (as well as surfaces with curvature) consisting of homogeneous materials characterized by arbitrary (linear, bianisotropic) constitutive equations. As was shown, treatment in the spectral domain allows constructing exact boundary conditions, often expressible in an analytic form. However it is not always possible to obtain an explicit form of the impedance boundary conditions in the space domain. For this reason it is necessary to derive an approximate representation of the impedance boundary conditions in the spectral domain and then apply the inverse Fourier transform. Generally, rational functions are used for approximate representations. This method has allowed deriving high-order impedance boundary conditions for various kinds of coatings including multi layered structures [9-11], inhomogeneous dielectric layers [12], multilayered coatings on curvilinear conductive bodies [13], structures composed of inhomogeneous dielectric and homogeneous bianisotropic layers [14], as well as for complex geometries [15]. Note that derivatives are applied to the tangential field components along the normal to the surface.

Another kind of the boundary conditions, which generalize the perfect boundaries and are mathematically no simpler than the impedance ones, was suggested in 2005 by Lindell and Sihvola [16,17]. These correspond to the perfectly electric-and-magnetic conducting (PEMC) boundary, viz.

$$\vec{H} + M\vec{E} = 0. \tag{3}$$

With $M = 0$ Eq. (3) corresponds to a perfectly electric conducting boundary, while with $M = \infty$ to a perfectly magnetic conducting one.

Despite that boundaries of this kind have a simple mathematical description, their physical implementation represents a difficult problem. A physically feasible model for the PEMC boundary was suggested in 2006 by Lindell [17], who showed that the PEMC boundary conditions in the case of vertical incidence can model the wave reflection from an anisotropic layer. Other physically feasible models have been also suggested, however the problem of wave diffraction by the boundary like that has not been considered.

A further generalization of the PEMC boundary is represented by the generalized soft-and-hard surface (GSHS) model treated in paper [18] in the form of the boundary conditions as follows

$$\vec{a} \cdot \vec{E} = 0 \text{ and } \vec{a} \cdot \vec{H} = 0,$$

where $\vec{a}$ and $\vec{b}$ are complex vectors satisfying the conditions $\vec{n} \cdot \vec{a} = \vec{n} \cdot \vec{b} = 0$ and $\vec{a} \cdot \vec{b} = 1$.

The GSHS can transform any specified polarization of an incident plane wave into any other polarization of the reflected wave with the properly selected vectors $\vec{a}$ and $\vec{b}$ [18].

Application of fractional derivatives results in fractional boundary conditions generalizing the perfect boundaries, viz.

$$D_n^{\alpha} U(\vec{r})\,|_S = 0, \tag{4}$$

where the fractional derivative is derived along the normal to the surface. The function U describes the electric or magnetic field tangential component. The fractional derivative is determined through the Riemann-Liouville integral over the semi-infinite interval [19].

The fractional order α assumes values lying between 0 and 1, The extreme magnitudes of the fractional order, $\alpha = 0$ and $\alpha = 1$, correspond to the perfectly electric conducting and perfectly magnetic conducting boundaries, respectively. In 2003 Veliev and Engheta [3,20,21] used the fractional boundary conditions for solving some problems of wave reflection. In particular, reflection factors from the structures described by fractional boundary conditions were presented. It was shown that the boundary is characterized by the absolute value of the reflection factor equal to 1, i.e., corresponds to the perfectly reflecting boundary, whereas the phase of the reflection factor is determined by the fractional order.

The fractional boundary conditions can be used for modeling the electromagnetic wave reflection from a certain kind of surfaces. Among principal problems associated with introducing the fractional boundary conditions one is determining the fractional order magnitude through the initial parameters of the problem. In the present paper the fractional boundary conditions are compared with the familiar impedance ones.

The fractional boundary conditions provide an example of the nonlocal boundary conditions. This means that the value of the function on the boundary is dependent on the fields at points located at a finite distance from the boundary, in contrast to the classical boundary conditions (perfectly electric conducting, perfectly magnetic conducting or impedance ones), where the magnitude on the boundary is solely determined by the fields at the points located infinitely near the boundary. This is associated with application of fractional order derivatives instead of ordinary ones.

The nonlocal boundary conditions in the scattering problems are widely for constructing numerical algorithms based on the finite element or finite difference methods [22,23]. The procedure implies analyzing a finite region which envelopes the scattering object, in order to restrict the space of computations. In this case the new boundary conditions (usually nonlocal ones) should be satisfied on the new region boundary. The nonlocal boundary conditions were used for the parabolic wave equation [23]. They represent a powerful alternative to the conventional treatment of absorbing layers [23].

Let us define the basic aspects associated with application of the fractional boundary conditions in the diffraction problems. These include i) development of a numerical-analytical solution method which would allow obtaining numerical results within a required accuracy to analyze characteristics of the fractional solution; ii) qualitative analysis of physical properties of the fractional boundary; iii) establishment of the relation between the fractional boundary and other familiar boundaries; and iv) physical implementation of the fractional boundary.

The problem solution of diffraction by structures with fractional boundary conditions requires a proper mathematical tool to be developed. The writers of papers [24,25] considered a 2D problem of wave diffraction by a strip characterized by fractional boundary conditions and presented the respective solution method based on the use of the fractional Green function and orthogonal polynomials.

In view of the fact that the mathematical tool of fractional derivatives has been developed, the fractional boundary conditions represent a simple generalization of the perfect boundaries (perfectly electric and perfectly magnetic conducting boundaries).

With account of the specific properties of the structures described by fractional boundary conditions it seems important to specify and implement structures of the kind. The fractional boundary conditions are required to be analyzed in detail from physical and mathematical points of view.

2. APPLICATION OF THE FRACTIONAL CURL OPERATOR TO SOLVING THE REFLECTION PROBLEM

Consider the classical 2D reflection problem for a plane wave incident obliquely (at an angle φ) upon an interface. The interface is located within the plane $y=0$ and is specified by boundary conditions with an impedance η, viz.

$$\vec{n}\times\vec{E}=\frac{\eta}{\eta_0}\vec{n}\times\left(\vec{n}\times\vec{H}\right),\ \text{with}\ y\to+0. \tag{5}$$

Assuming that the solution $\left(\vec{E}^0,\vec{H}^0\right)$ of the problem for the impedance boundary Eq. (5) is known, we will consider two ways of constructing the fractional field $\left(\vec{E}^\alpha,\vec{H}^\alpha\right)$. These are i) application of the curl^α operator to the total field, viz.

$$\left(\vec{E}^\alpha,\vec{H}^\alpha\right)\equiv(ik_0)^{-\alpha}\text{curl}^\alpha\left(\vec{E}^0,\vec{H}^0\right); \tag{6}$$

and ii) application of the curl^α operator to the reflected field, viz.

$$\left(\vec{E}^{\alpha,r},\vec{H}^{\alpha,r}\right)\equiv(ik_0)^{-\alpha}\text{curl}^\alpha\left(\vec{E}^{0,r},\vec{H}^{0,r}\right)\ \text{and}\ \left(\vec{E}^{\alpha,i},\vec{H}^{\alpha,i}\right)\equiv\left(\vec{E}^{0,i},\vec{H}^{0,i}\right). \tag{7}$$

In the first case (Eq. (6)) the fractional boundary occupies an intermediate position between an impedance boundary with an impedance η ($\alpha=0$) and a dual boundary with an impedance η^{-1} ($\alpha=1$). The fractional boundary can be described using anisotropic impedance boundary conditions, viz.

$$\vec{n}\times\vec{E}^\alpha=\overline{\eta}_\alpha\vec{n}\times\left(\vec{n}\times\vec{H}^\alpha\right),$$

where the impedance $\overline{\eta}_\alpha=\begin{pmatrix}\eta_{11}^\alpha & \eta_{12}^\alpha\\ \eta_{21}^\alpha & \eta_{22}^\alpha\end{pmatrix}$ is a tensor.

In particular cases of a tensorial $\overline{\eta}_\alpha$ expressions have been derived which determine $\overline{\eta}_\alpha$ via η and α. Assuming that $\eta_{21}^\alpha = \eta_{12}^\alpha = 0$ we can obtain expressions for the tensor elements of the fractional impedance $\overline{\eta}_\alpha$,

$$\eta_{11}^\alpha / \eta_0 = -\frac{1}{\sin\varphi}\frac{e^{i\pi\alpha}(B-A)+BR_E+AR_H}{e^{i\pi\alpha}(B-A)-(BR_E+AR_H)}$$

and

$$\eta_{22}^\alpha / \eta_0 = -\sin\varphi\frac{e^{i\pi\alpha}(A+B)+AR_E-BR_H}{e^{i\pi\alpha}(A+B)-(AR_E-BR_H)},$$

where $A=\sin(\pi\alpha/2)$, $B=\cos(\pi\alpha/2)$. Here $R_E = -\dfrac{1-\eta/\eta_0\sin\varphi}{1+\eta/\eta_0\sin\varphi}$ and $R_H = -\dfrac{1-(\eta/\eta_0)^{-1}\sin\varphi}{1+(\eta/\eta_0)^{-1}\sin\varphi}$ are the reflection factors.

If the initial impedance is zero ($\eta = 0$), then

$$\eta_{11}^\alpha / \eta_0 = \frac{i}{\sin\varphi}\tan(\pi\alpha/2) \tag{8}$$

and

$$\eta_{22}^\alpha / \eta_0 = i\sin\varphi\tan(\pi\alpha/2). \tag{9}$$

Eqs. (8) and (9) generalize the expression for the fractional impedance derived in paper [1] for the case of oblique incidence.
It can be shown that in the case of $\eta_{11}^\alpha = \eta_{22}^\alpha = 0$ the surface currents are related as follows

$$j_{ex}^\alpha = -\eta_{12}^\alpha j_{mx}^\alpha \text{ and } j_{ez}^\alpha = -\eta_{21}^\alpha j_{mz}^\alpha .$$

The surface currents are oriented in parallel to each other in contrast to the case of the isotropic impedance boundary where the currents are perpendicular. Boundaries of the kind were analyzed in papers by Lindell and Sihvola with the aim of describing the PEMC boundary Eq. (3).

To construct a model boundary for the case Eq. (7), let us consider a biisotropic layer ($0<y<L$) above the perfectly electric conducting boundary ($y=0$). The problem geometry is shown in Fig. 1. The biisotropic medium is described by the following constitutive equations

$$\vec{D} = \varepsilon_2 \vec{E} + \xi_2 \vec{H} \text{ and } \vec{B} = \zeta_2 \vec{E} + \mu_2 \vec{H},$$

where ε_2 and μ_2 are the permittivity and permeability, respectively. The coefficients ξ_2 and ζ_2 are expressible as

$$\xi_2 = (\chi_2 - i\kappa_2)\sqrt{\varepsilon_0 \mu_0} \text{ and } \zeta_2 = (\chi_2 + i\kappa_2)\sqrt{\varepsilon_0 \mu_0},$$

where χ_2 is the Tellegen parameter and κ_2 is the chirality.

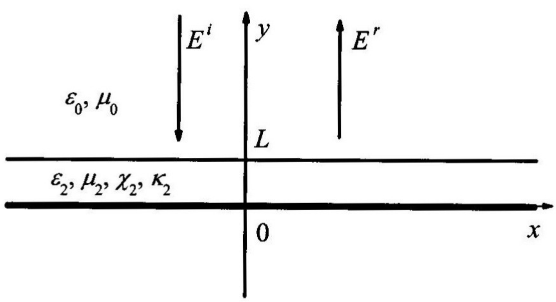

FIG. 1: Geometry of the problem of plane wave reflection from a biisotropic layer

The reflection factor from a fractional boundary, $\overline{R}^{\alpha}$, can be expressed through the initial ones R_E and R_H as

$$\vec{E}^{\alpha,r} = \overline{R}^{\alpha} \vec{E}^{\alpha,i}, \text{ with } \overline{R}^{\alpha} = \begin{pmatrix} -BR_H & AR_E \\ AR_H & BR_E \end{pmatrix}.$$

Application of the fractional curl operator results in the field polarization reversal. The effect of polarization reversal is also observed in the case of reflection from or transmission through a biisotropic layer. The fractional boundary can be used for modeling a biisotropic layer [3] provided that α has been selected such that the co-polarization R_{co} and cross-polarization R_{cr} factors were related as

$$R_{co} = -\cos(\pi\alpha/2) \text{ and } R_{cr} = \sin(\pi\alpha/2).$$

The equation for determining α takes the form

$$\tan\left(\frac{\pi\alpha}{2}\right) = -\frac{R_{cr}}{R_{co}} = -\frac{2\eta_0\eta_2 \sin\vartheta_2 \sin^2(k_2 L\cos\vartheta_2)}{(\eta_0^2 - \eta_2^2)\sin^2(k_2 L\cos\vartheta_2) - \eta_0^2\cos^2\vartheta_2},$$

where $\eta_2 = \sqrt{\mu_2 / \varepsilon_2}$, $\chi_2 = \sin \vartheta_2$ and $k_2 = \omega\sqrt{\varepsilon_2\mu_2}$.

Figures 2 and 3 present dependences of the fractional order and the reflection factors versus the layer parameters. Values of α close to 1 correspond to $R_{co} \to 0$ and $R_{cr} \to 1$, which case corresponds to the polarization reversal effect where the incident plane wave is transformed into the reflected plane wave with the polarization turned by 90° with respect to the incident one.

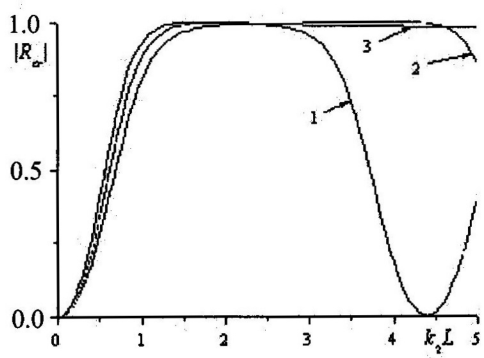

FIG. 2: Fractional order α as a function of the layer thickness k_2L for $\eta_2/\eta_0 = 0.8$. Curves 1, 2 and 3 correspond to $\chi_2 = 0.7$, $\chi_2 = 0.85$ and $\chi_2 = 1$, respectively

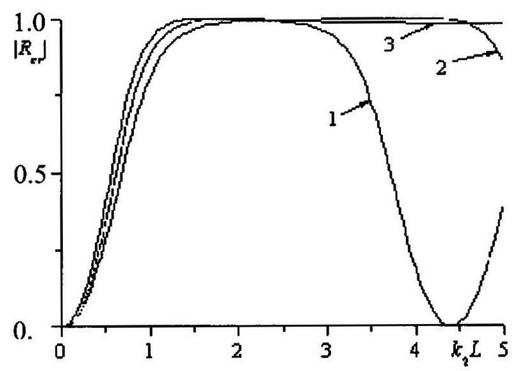

FIG. 3: Cross-polarization reflection factor R_{cr} as a function of the layer thickness k_2L for $\eta_2/\eta_0 = 0.8$. Curves 1, 2 and 3 correspond to $\chi_2 = 0.7$, $\chi_2 = 0.85$ and $\chi_2 = 1$, respectively

3. FRACTIONAL BOUNDARY CONDITIONS IN THE PROBLEMS OF WAVE REFLECTION FROM AN INTERFACE

The reflection factor of a plane wave from a semi-infinite half-space $(y < 0)$ with the fractional boundary conditions in the form

$$D_{ky}^{\alpha} E_z(x, y) = 0, \; y \to +0 \tag{10}$$

can be expressed as

$$R_\alpha = -(-1)^\alpha = -e^{-i\pi\alpha}. \tag{11}$$

In the case of an impedance boundary the reflection factor is

$$R_{imp} = -\frac{1-\eta/\eta_0 \sin\varphi}{1+\eta/\eta_0 \sin\varphi}, \qquad (12)$$

where φ is the angle of incidence.

Comparison of the reflection factors Eq. (11) and Eq. (12) allows deriving the relation between the fractional order and the impedance. Hence, the fractional boundary conditions can model the impedance boundary conditions if the fractional order is selected from the relation as follows [24]

$$\alpha = \frac{1}{i\pi}\ln\frac{1+\eta/\eta_0 \sin\varphi}{1-\eta/\eta_0 \sin\varphi}. \qquad (13)$$

This relation was also presented in paper [21].

The fractional order $\alpha = 0$ corresponds to the perfectly electric conducting boundary ($\eta = 0$). The value $\alpha = 1$ corresponds to the impedance $\eta = i\infty$, which case is associated with the perfectly magnetic conducting boundary.

Consider the reflection problem for a wave obliquely incident upon a dielectric layer. The problem geometry is shown in Fig. 4. The dielectric layer can be described by the fractional boundary conditions in the form of Eq. (10) (with $y = L+0$) provided that α has been selected from the relation

$$\alpha = \alpha(\varepsilon, \mu, L) = \frac{1}{i\pi}\ln\frac{R_{12} + R_{23}e^{-2ik_2L}}{1+R_{12}R_{23}e^{-2ik_2L}}, \qquad (14)$$

where $R_{ij} = (\eta_j - \eta_i)/(\eta_j + \eta_i)$, with $\eta_j = \sqrt{\mu_j/\varepsilon_j}$, $i, j = 1,2,3$.

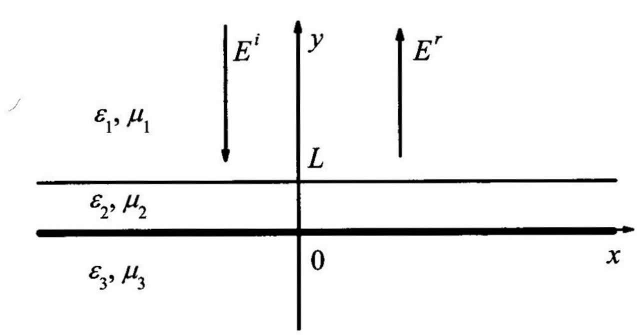

FIG. 4: Problem geometry of plane wave reflection from a dielectric layer

If $|R_{12}| \to 0$ and $|R_{23}| \to 0$, then Eq. (14) yields

$$\alpha = \frac{1}{i\pi} \ln\left(R_{12} + R_{23} e^{-2ik_2 L}\right).$$

Whereas if medium 3 is a perfectly electric conducting one, that is $\varepsilon_3 = \infty$, then $R_{23} = -1$ and the reflection factor can be expressed as

$$R = \frac{\eta_2' + i\cot(k_2 L)}{\eta_2' - i\cot(k_2 L)}.$$

In this case the fractional order is

$$\alpha = \frac{1}{i\pi} \ln \frac{\eta_2' + i\cot(k_2 L)}{\eta_2' - i\cot(k_2 L)},$$

where $\eta_2' = \eta_2 / \eta_1$ is the normalized impedance of medium 2.

Let us recast the equation in the form

$$\cot(k_2 L) = \eta_2' \cot(\pi\alpha / 2),$$

$$k_2 L = \cot^{-1}(\eta_2' \cot(\pi\alpha / 2)).$$

The value $\alpha = 0$ corresponds to the case where $\cot(k_2 L) = \infty$ and $R = -1$. This is met in the obvious situation where the layer thickness is $L = 0$, as well for $k_2 L = \pi n$, $n = 1, 2, 3, \ldots$. With $\alpha = 1$ we have $\cot(k_2 L) = 0$ ($k_2 L = \pi / 2 + \pi n$, $n = 0, 1, 2, \ldots$) and $R = 1$.

4. WAVE DIFFRACTION BY A STRIP WITH FRACTIONAL BOUNDARY CONDITIONS

Consider the 2D problem of wave diffraction by a strip with fractional boundary conditions in the form

$$D_{ky}^{\alpha} E_z(x, y) = 0, \text{ with } y \to \pm 0. \quad (15)$$

The solution method of the problem in the case of E-polarization was considered in detail in paper [24]. The writers presented numerical results for the scattering characteristics of the strip, including the scattering pattern, backscatter cross-section area and surface current density distribution.

The fractional boundary can support both the electric and magnetic currents. A similar distribution of surface currents is observed in the case of the impedance surface. Paper [25] presents the results of qualitative and quantitative comparison between the scattering characteristics of a strip with fractional boundary conditions and those of an impedance strip. In the latter case the impedance was determined from the relation Eq. (13) which is exact provided the boundaries are infinite. The obtained relationship can be used in consideration of finite boundaries in the same manner as the Leontovich boundary conditions were first introduced for infinite boundaries to determine the impedance value, and then were applied to finite boundaries. In this sense the fractional boundary conditions are approximate similar to the impedance ones. The error arising from the use of the fractional boundary conditions instead of the impedance ones in the case of finite boundaries is to be additionally analyzed and can be estimated numerically, in particular, through solving the respective diffraction problems. The range of applicability of the fractional boundary conditions for modeling the impedance ones was analyzed in paper [25] by way of example of the problem of wave diffraction by a strip. Comparing the far-fields for the strip with the fractional boundary conditions and for the impedance strip within the physical optics approximation it is possible to derive an analytic equation relating the fractional order and the impedance [25]. This equation exactly coincides with the expression Eq. (13) obtained for infinite boundaries.

The fractional boundary conditions in the diffraction problems are similar to the impedance boundary conditions [25] in the case of sufficiently short wavelengths when the edge effects are less important (see Fig. 5).

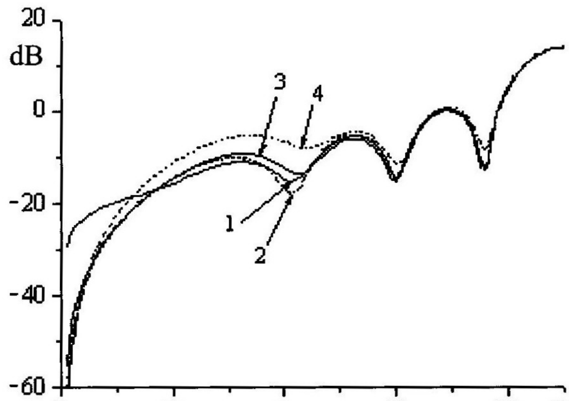

FIG. 5: Backscatter cross-section areas calculated with $ka = 2\pi$ for a fractional strip with $\alpha = 0.25$ (curve 1), an impedance strip with $\eta = \eta(0.25)$ (curve 2), a fractional strip with $\alpha = 0.75$ (curve 3) and an impedance strip with $\eta = \eta(0.75)$ (curve 4)

However, in the case of great wavelengths structures with the fractional boundary conditions show some specific features. As it was shown in paper [24], the backscatter cross-section of a strip characterized by fractional boundary conditions with the fractional order $\alpha = 1/2$ demonstrates a resonance behavior in dependence on the angle of incidence if the wavelength is greater than the transverse dimension of the strip (see Fig. 6). No resonance effect like that is observed in the case of an impedance strip of the same width. The resonance effect is associated with the properties of the surface itself, rather than with the strip geometry. The strip with the fractional boundary conditions of order $\alpha = 1/2$ shows the "superwave" properties. Note that the "superwave" properties like these are known for surfaces with fractal geometry. A possibility of application of the fractional boundary conditions to analyzing rough surfaces was mentioned by Potapov in paper [26].

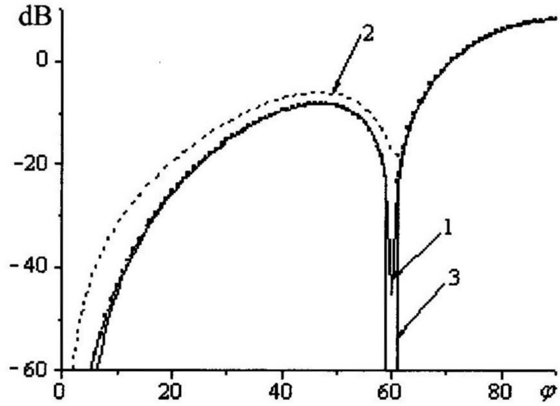

FIG. 6: Backscatter cross-section areas calculated numerically (curve 1) and analytically (curve 3) for a fractional strip with $\alpha = 1/2$, and an impedance strip with $\eta = -i/\sin\varphi$ (curve2)

5. PHYSICAL IMPLEMENTATION OF THE FRACTIONAL BOUNDARY CONDITIONS

Consider the relation Eq. (5). As can be seen, the fractional boundary conditions with $0 \le \alpha \le 1$ correspond to pure imaginary impedance $\eta = ia$ ($0 \le a < \infty$).

In a particular case of the fractional order equal to $\alpha = 1/2$ the fractional boundary conditions correspond to the impedance $\eta = -i$. If the impedance is defined as $\eta = \sqrt{\mu/\varepsilon}$, then the fractional boundary conditions can be simulated using a layer with the parameters related as

$$\frac{\varepsilon}{\varepsilon_0} + \frac{\mu}{\mu_0}\sin^2\varphi = 0.$$

The case $\mu / \mu_0 = 1$ corresponds to the value of the dielectric constant equal to $\varepsilon / \varepsilon_0 = -\sin^2 \varphi$.

If the fractional boundary conditions are applied to modeling a resistive layer [1] with the resistivity $R_e = -\frac{i}{k_0 \tau (\varepsilon - 1)}$, where the layer thickness τ is assumed to be sufficiently small $(k_0 \tau \to 0)$, then the following expression can be derived for determining the fractional order

$$-\frac{i}{\sin \varphi} \tan(\pi \alpha / 2) = -\frac{i}{k_0 \tau (\varepsilon - 1)}.$$

In a particular case of $\alpha = 1/2$ we have

$$k_0 \tau = -\frac{1}{\varepsilon / \varepsilon_0 - 1},$$

which corresponds to the permittivity $\varepsilon / \varepsilon_0 = 1 - \frac{1}{k_0 \tau}$ whose value is usually less than 0.

6. CONCLUSIONS

Basic possibilities are considered of applying fractional operators to describing the reflective properties of various surfaces. The fractional curl operator and fractional derivative are analyzed. The fractional field determined with the use of the fractional curl operator can describe the solution of the problem of wave reflection from a boundary characterized by anisotropic impedance boundary conditions or from a biisotropic layer. Relations have been derived for determining the order of the fractional curl operator trough constitutive parameters.

Along with the impedance boundary conditions, the fractional boundary conditions represent a generalization of the perfect boundaries (perfectly electric and perfectly magnetic conducting). Expressions have been obtained for determining the fractional order via the parameters of the structures represented by surfaces modeled by impedance boundary conditions, as well as by a resistive layer. By the example of the diffraction problem by strips, the conditions of applicability of the fractional boundary conditions to modeling impedance boundaries have been determined. The fractional boundary conditions can be used for modeling the scattering properties of boundaries for other surfaces.

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